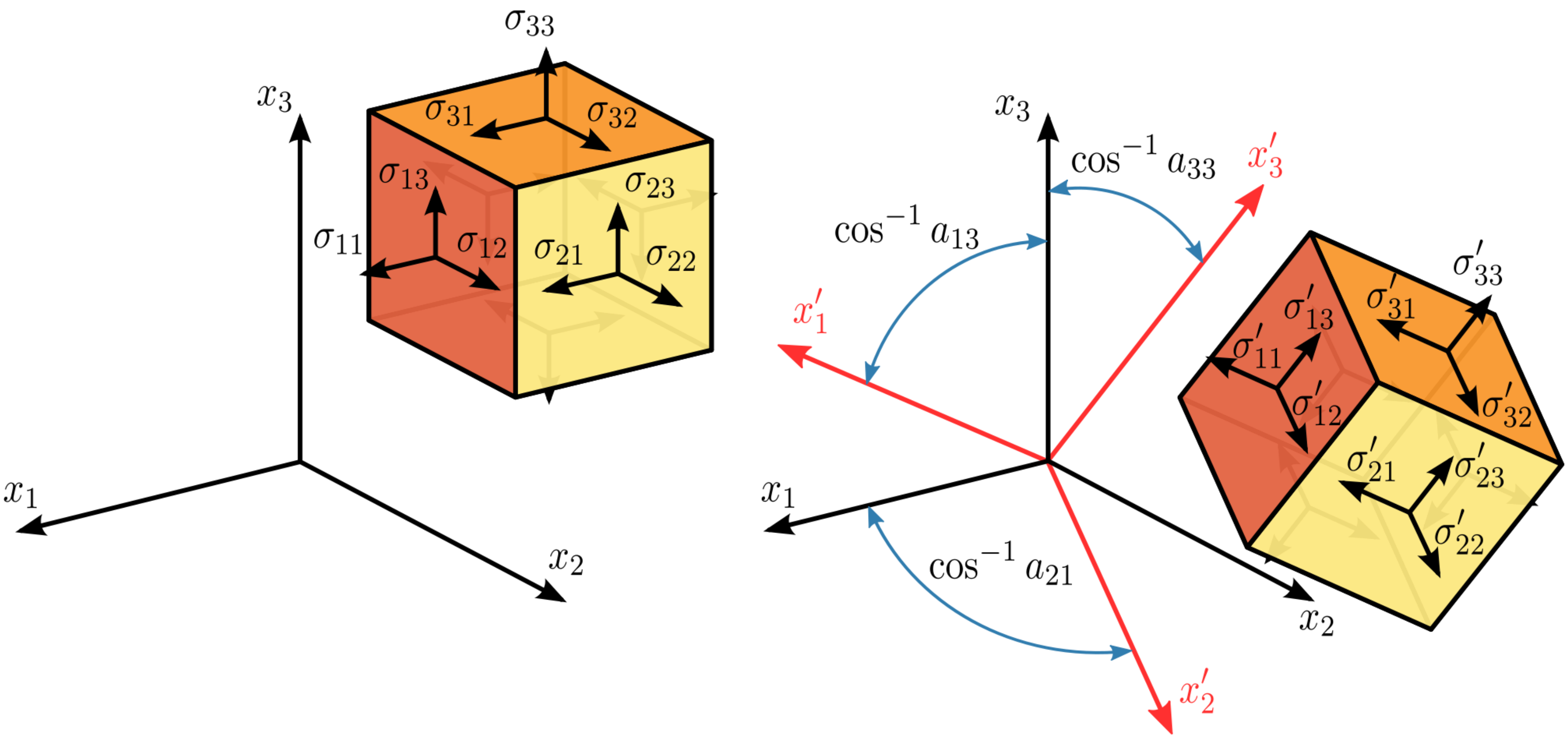
Mechanics of Solids-2

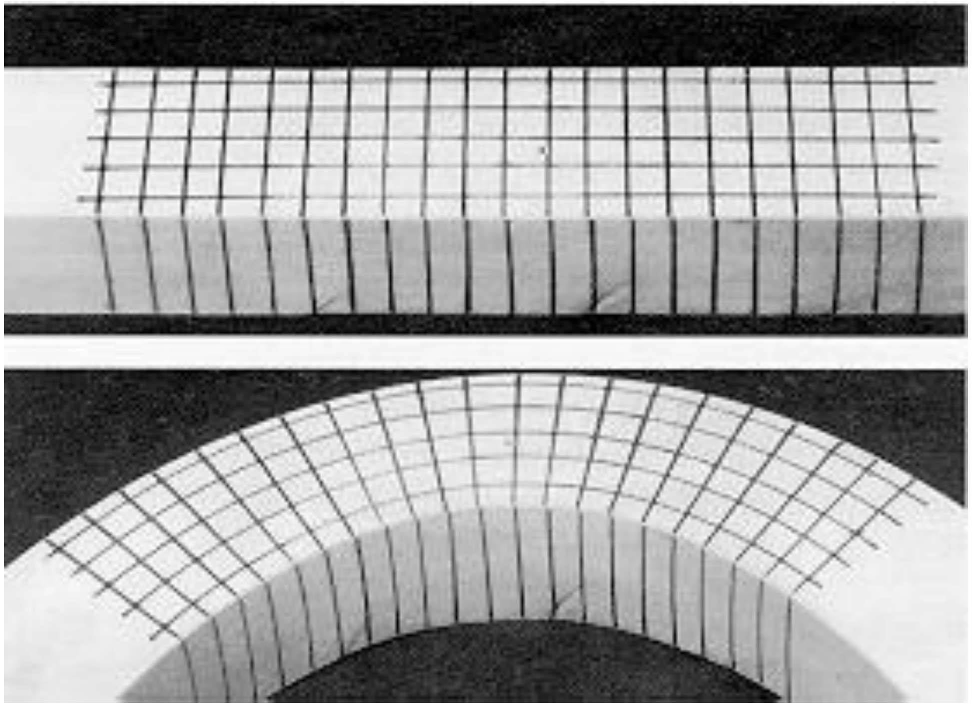
Lecture # 9



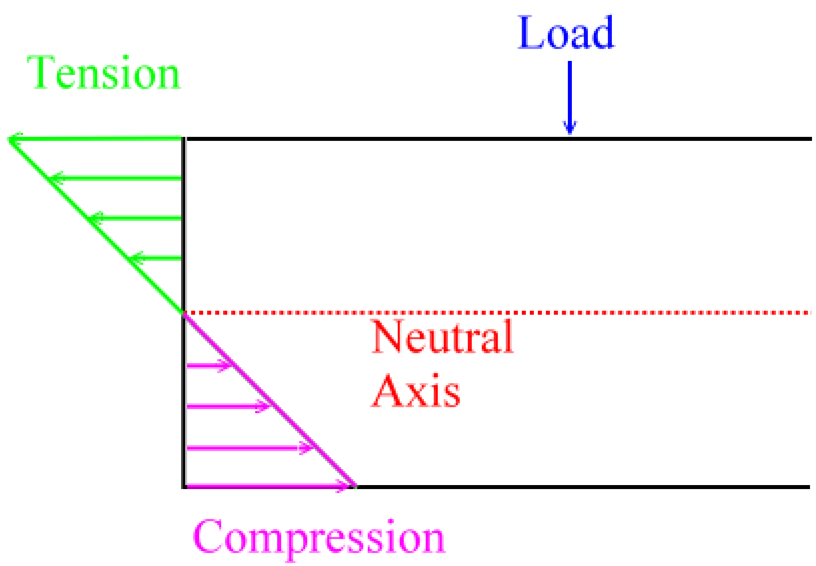
Shad Muhammad
Lecturer
COMSATS UNIVERSITY ISLAMBABAD, SAHIWAL CAMPUS.

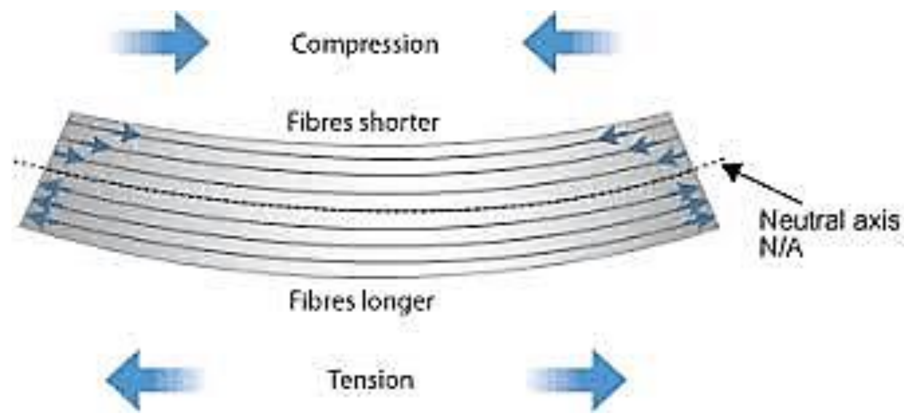
Learning Outcome

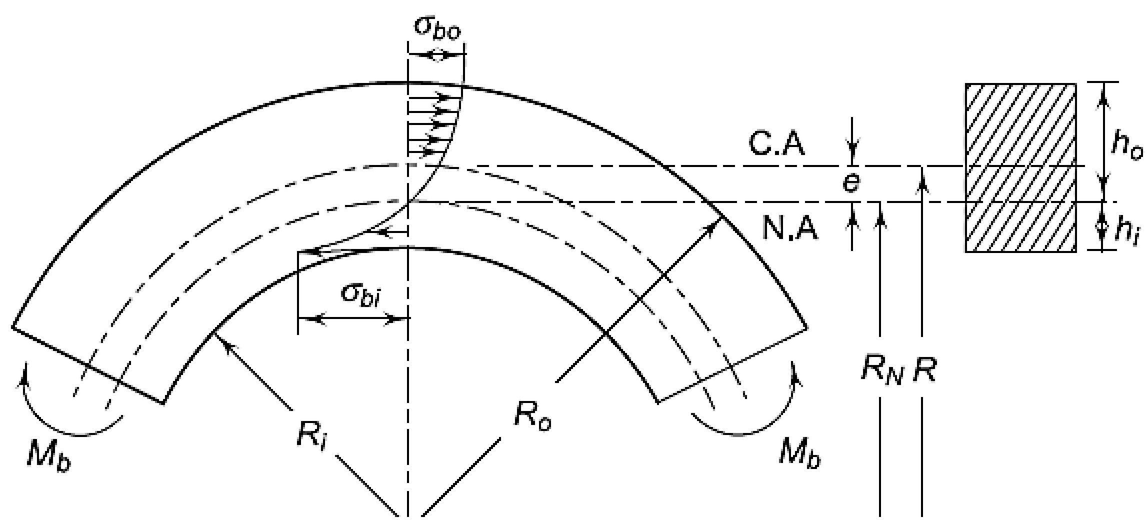
- **Bending about both Principal Axis**
- **Elastic Bending with Axial Loads**
- **Kern of Section**



Learning Outcome

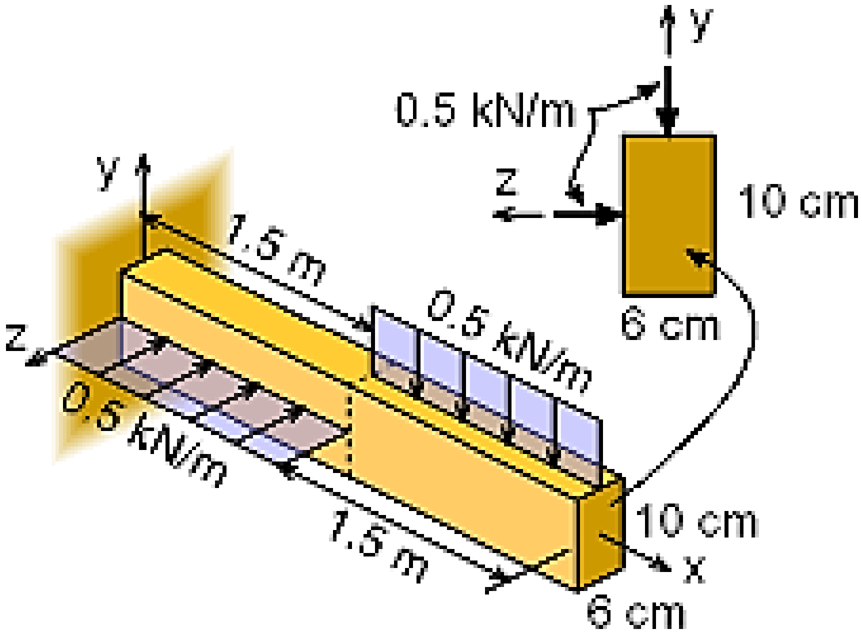






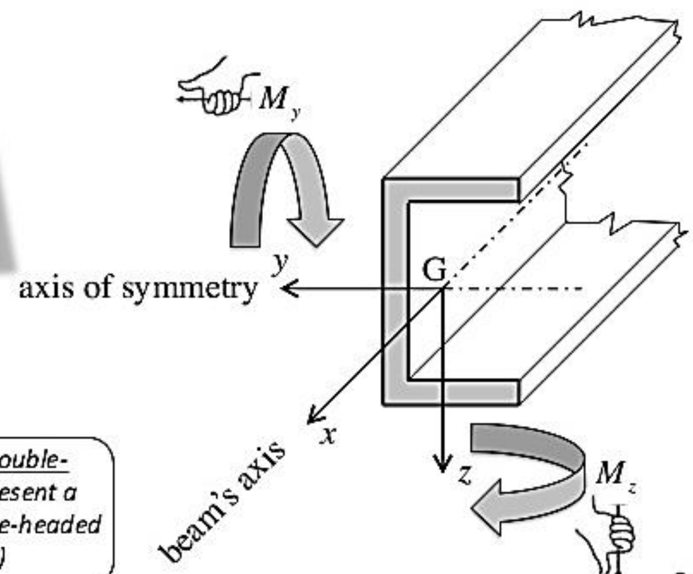
Stresses in Curved Beam (C.A. = centroidal axis; N.A. = neutral axis)

Bending about Both Principal Axis



Right-Hand Rule

If the thumb point to the positive direction of the axis, then the curling of the other fingers give the positive direction of the bending



Noteworthy: Sometimes a *__double-headed arrow__* *is used to represent a moment (as opposed to a single-headed arrow used for a force)*

Bending about Both Principal Axis

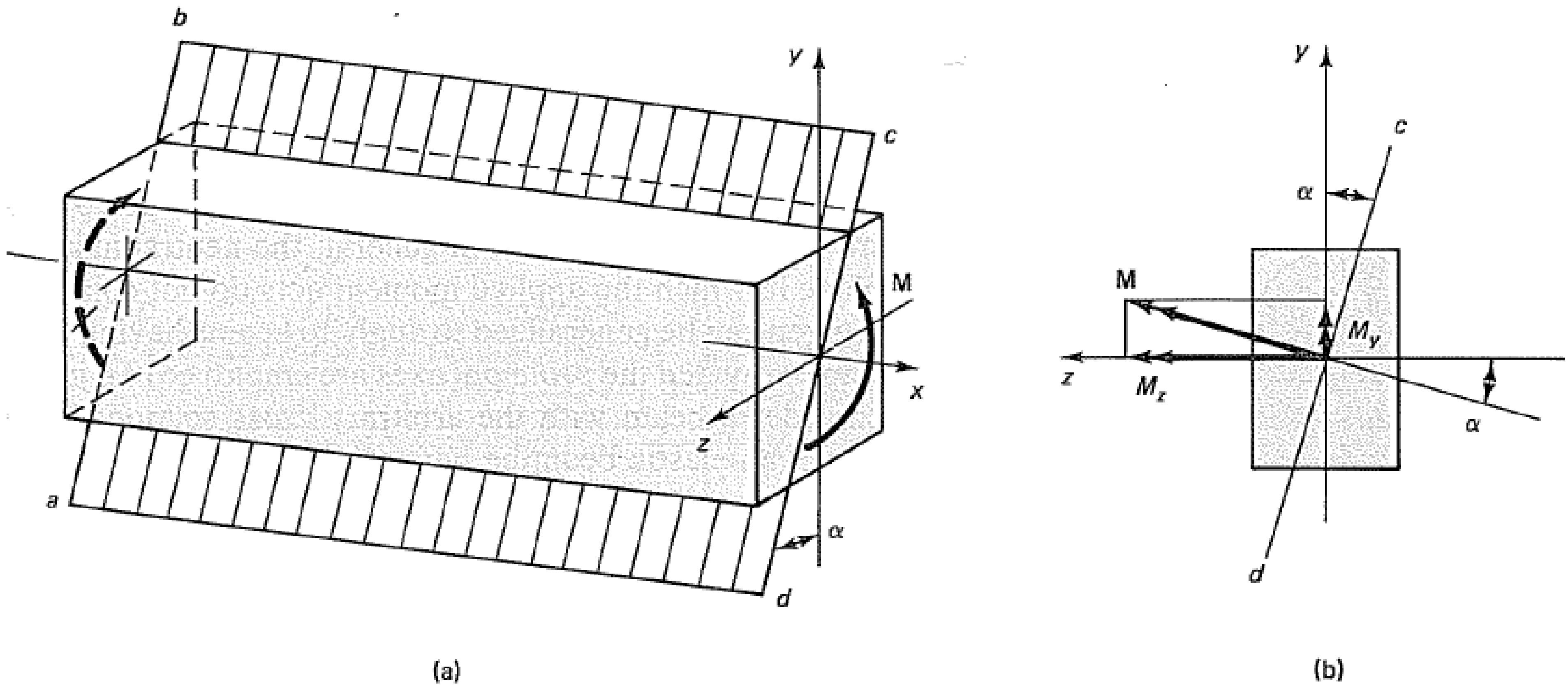


Fig. 6-33 Unsymmetrical bending of a beam with doubly symmetric cross section.

Bending about Both Principal Axis

As a simple example of skew or unsymmetrical pure bending, consider the rectangular beam shown in Fig. 6-33. The applied moments *M* act in the plane *abcd*. By using the vector representation for **M** shown in Fig. 6-33(b), this vector forms an angle α with the z axis and can be resolved into the two components, M_y and M_z. Since the cross section of this beam has symmetry about both axes, the formulas derived in Section 6-3 are directly applicable. Because of symmetry, the product of inertia for this section is zero, and the orthogonal axes shown are the *principal* axes for the cross section. This also holds true for the centroidal axes of singly symmetric areas. (For details see Sections 6-15 and 6-16.)

By assuming *elastic* behavior of the material, a superposition of the stresses caused by M_y and M_z is the solution to the problem. Hence, using Eqs. 6-11 and 6-12,

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{6-41}$$

where all terms have the previously defined meanings.

Bending about Both Principal Axis

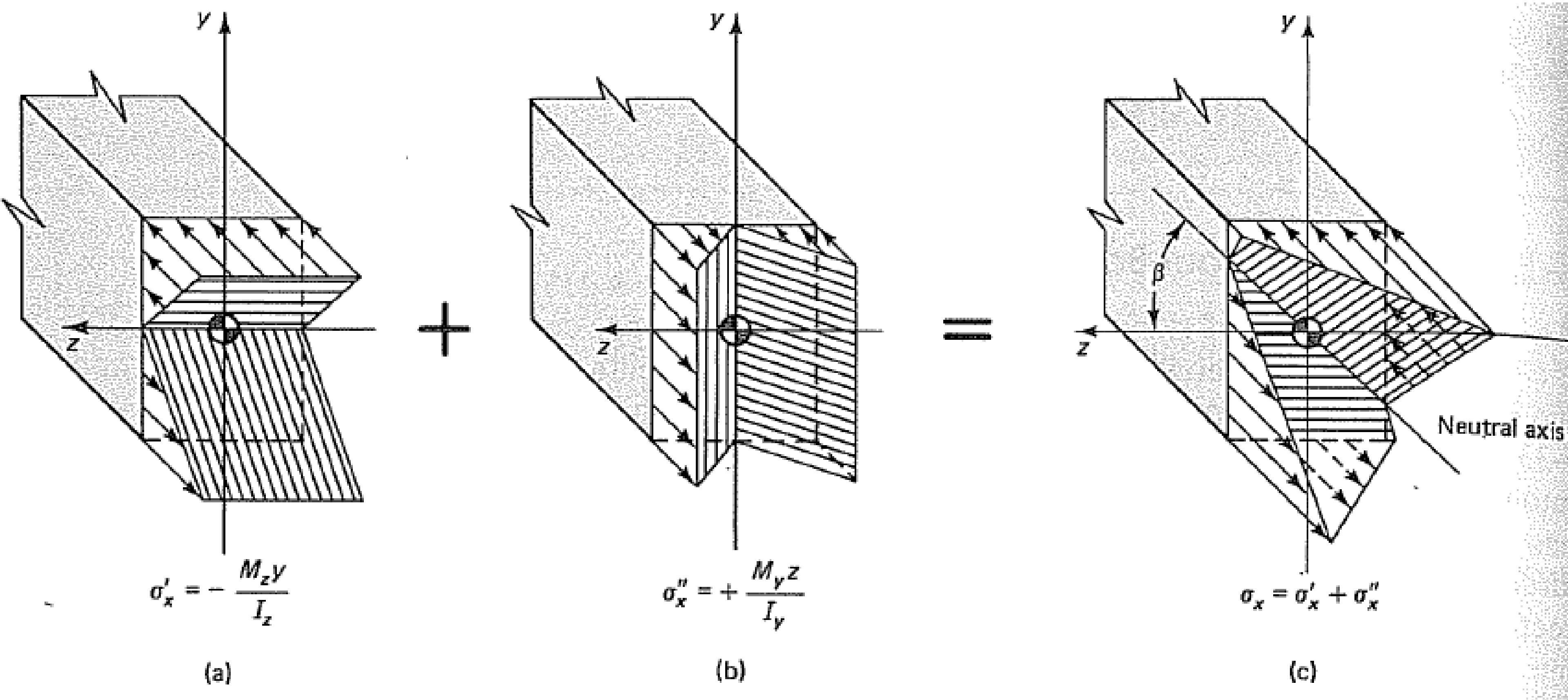


Fig. 6-34 Superposition of elastic bending stresses.

Bending about Both Principal Axis

A graphical illustration of superposition is given in Fig. 6-34. Note that a line of zero stress, i.e., a neutral axis, forms at an angle β with the z axis. Analytically, such an axis can be determined by setting the stress given by Eq. 6-41 to zero, i.e.,

$$-\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0 \qquad \text{or} \qquad \tan\beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y} \tag{6-42}$$

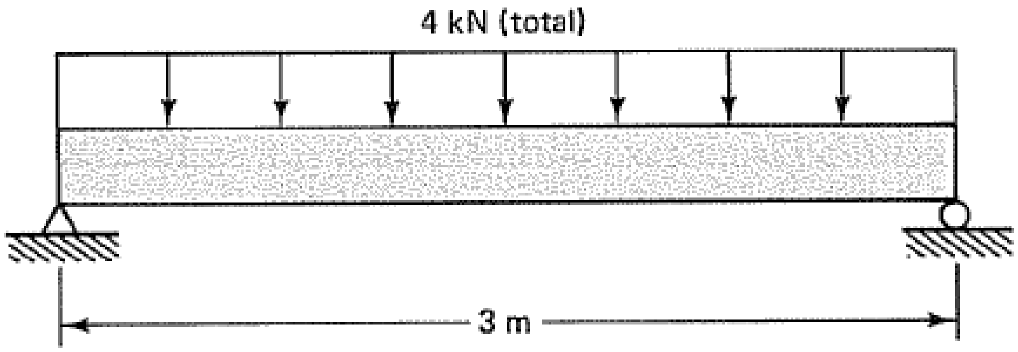
Since, in general, $M_y = M \sin\alpha$ and $M_z = M \cos\alpha$, this equation reduces to

$$\tan\beta = \frac{I_z}{I_y}\tan\alpha \tag{6-43}$$

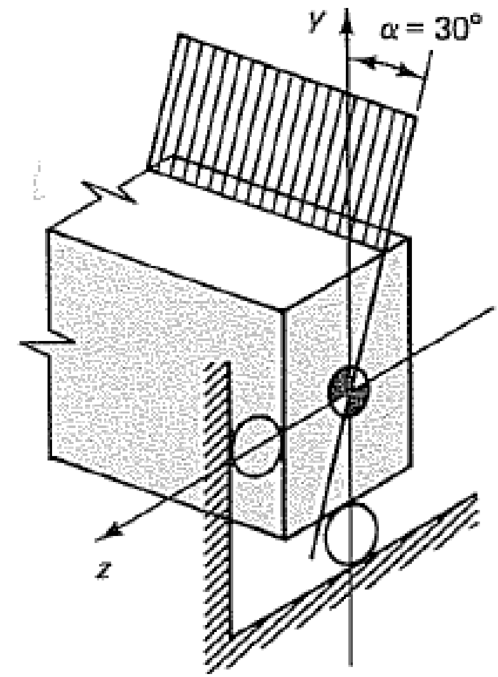
This equation shows that unless $I_z = I_y$, or α is either 0° or 90°, the angles α and β are not equal. Therefore, in general, the neutral axis and the normal to a plane in which the applied moment acts do not coincide.

Bending about Both Principal Axis – Example Problem

The 100 by 150 mm wooden beam shown in Fig. 6-36(a) is used to support a uniformly distributed load of 4 kN (total) on a simple span of 3 m. The applied load acts in a plane making an angle of 30° with the vertical, as shown in Fig. 6-36(b) and again in Fig. 6-36(c). Calculate the maximum bending stress at midspan, and, for the same section, locate the neutral axis. Neglect the weight of the beam.



(a)



(b)

Solution

The maximum bending *in the plane of the applied load* occurs at midspan, and according to Example 5-8, it is equal to $w_o L^2/8$ or $WL/8$, where W is the total load on span L. Hence,

$$M = \frac{WL}{8} = \frac{4 \times 3}{8} = 1.5 \text{ kN}\cdot\text{m}$$

Next, this moment is resolved into components acting around the respective axes, and I_z and I_y are calculated.

$$M_z = M \cos \alpha = 1.5 \times \sqrt{3}/2 = 1.3 \text{ kN}\cdot\text{m}$$
$$M_y = M \sin \alpha = 1.5 \times 0.5 = 0.75 \text{ kN}\cdot\text{m}$$
$$I_z = 100 \times 150^3/12 = 28.1 \times 10^6 \text{ mm}^4$$
$$I_y = 150 \times 100^3/12 = 12.5 \times 10^6 \text{ mm}^4$$

Bending about Both Principal Axis – Example Problem

By considering the sense of the moment components, it can be concluded that the maximum tensile stress occurs at A. Similar reasoning applies when considering the other corner points. Alternatively, the values for the coordinate points can be substituted directly into Eq. 6-41. On either basis,

$$\sigma_A = -\frac{M_z(-c_1)}{I_z} + \frac{M_y c_2}{I_y} = \frac{1.3 \times 10^6 \times 75}{28.1 \times 10^6} + \frac{0.75 \times 10^6 \times 50}{12.5 \times 10^6}$$

$$= +3.47 + 3.00 = +6.47 \text{ MPa}$$

$$\sigma_B = +3.47 - 3.00 = +0.47 \text{ MPa}$$

$$\sigma_C = -3.47 - 3.00 = -6.47 \text{ MPa}$$

$$\sigma_D = -3.47 + 3.00 = -0.47 \text{ MPa}$$

Note that the stress magnitudes on diametrically opposite corners are numerically equal.

The neutral axis is located by the angle β, using Eq. 6-43:

$$\tan\beta = \frac{28.1 \times 10^6}{12.5 \times 10^6} \tan 30° = 1.30 \qquad \text{or} \qquad \beta = 52.4°$$

Alternatively, it can be found from the stress distribution, which varies linearly between any two points. For example, from similar triangles, $a/(150 - a) = 0.47/6.47$, giving $a = 10.2$ mm. This locates the neutral axis shown in Fig. 6-36(e) as it must pass through the section centroid. These results lead to the same β.

Bending about Both Principal Axis – Example Problem

$$\sigma_A = -\frac{M_z(-c_1)}{I_z} + \frac{M_y c_2}{I_y} = \frac{1.3 \times 10^6 \times 75}{28.1 \times 10^6} + \frac{0.75 \times 10^6 \times 50}{12.5 \times 10^6}$$

$$= +3.47 + 3.00 = +6.47 \text{ MPa}$$

$$\sigma_B = +3.47 - 3.00 = +0.47 \text{ MPa}$$

$$\sigma_C = -3.47 - 3.00 = -6.47 \text{ MPa}$$

$$\sigma_D = -3.47 + 3.00 = -0.47 \text{ MPa}$$

Note that the stress magnitudes on diametrically opposite corners are numerically equal.

The neutral axis is located by the angle β, using Eq. 6-43:

$$\tan \beta = \frac{28.1 \times 10^6}{12.5 \times 10^6} \tan 30° = 1.30 \qquad \text{or} \qquad \beta = 52.4°$$

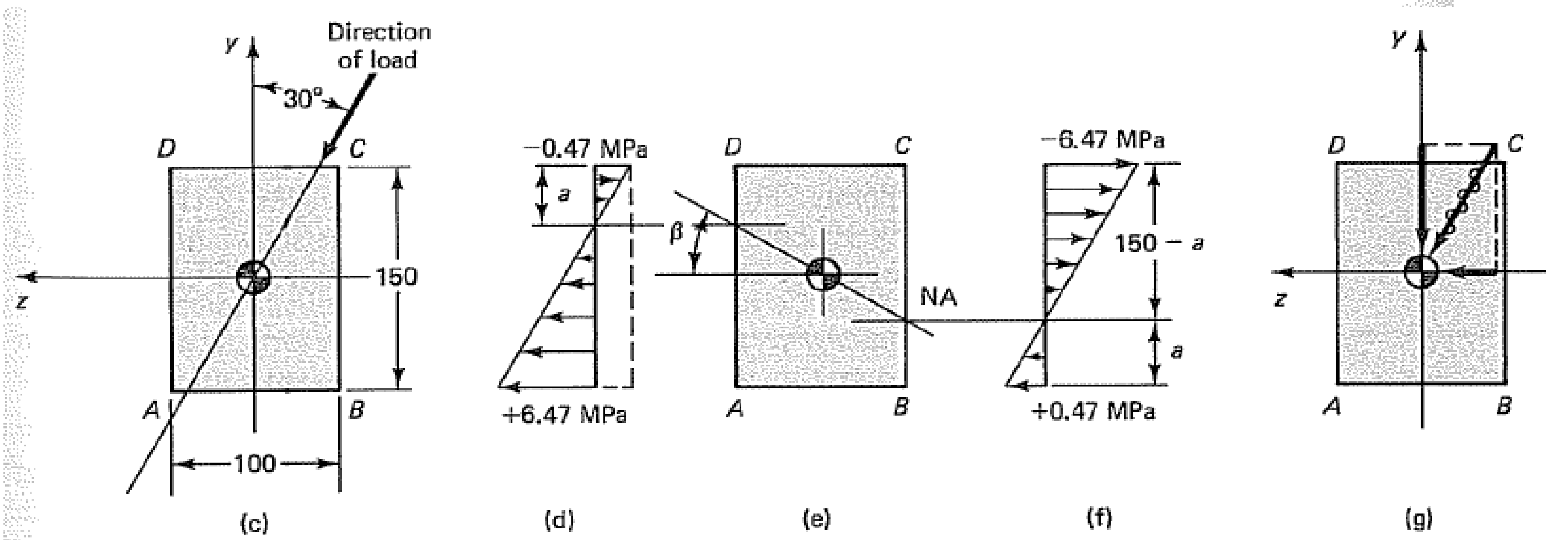


Fig. 6-36

Elastic Bending with Axial Loads

A solution for pure bending around both principal axes of a member can be extended to include the effect of axial loads by employing *superposition*. Such an approach is applicable only in the range of *elastic* behavior of members. Further, if an applied axial force causes compression, a member must be stocky, lest a buckling problem of the type considered in Chapter 11 arises. With these reservations, Eq. 6-41 can be generalized to read

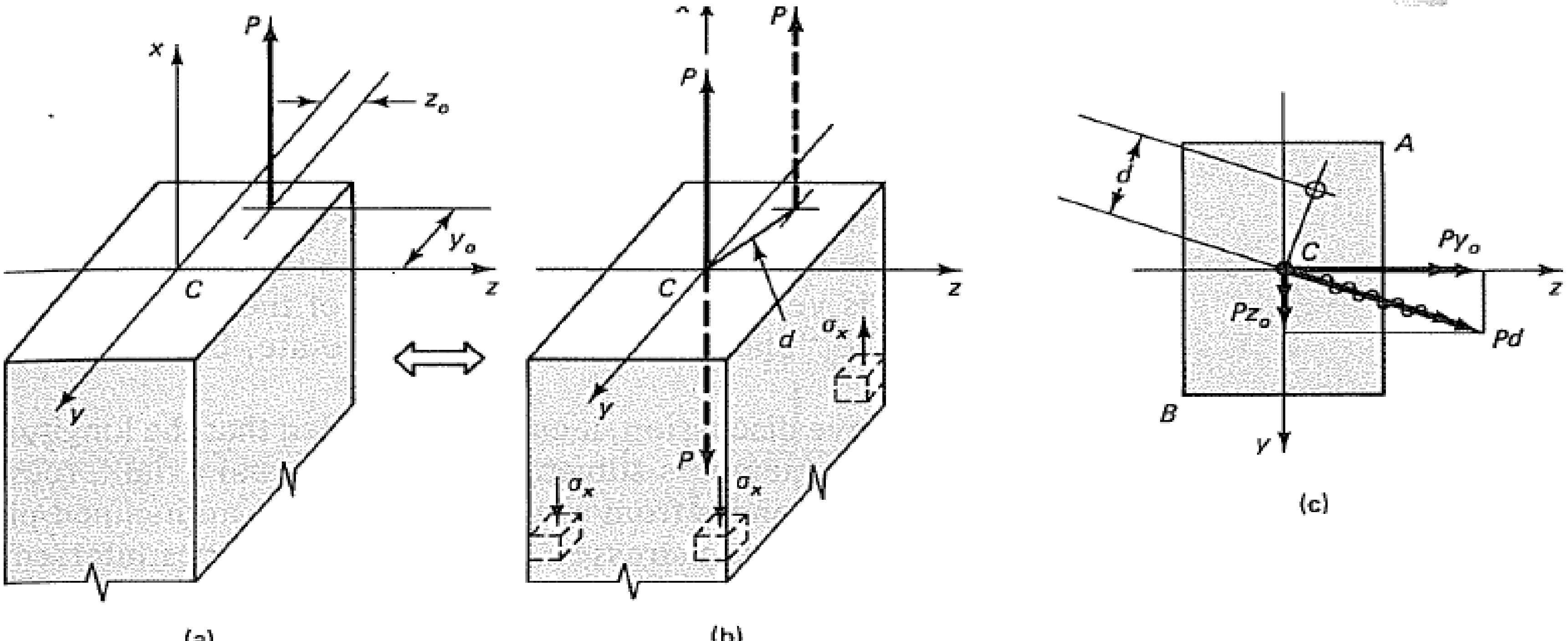
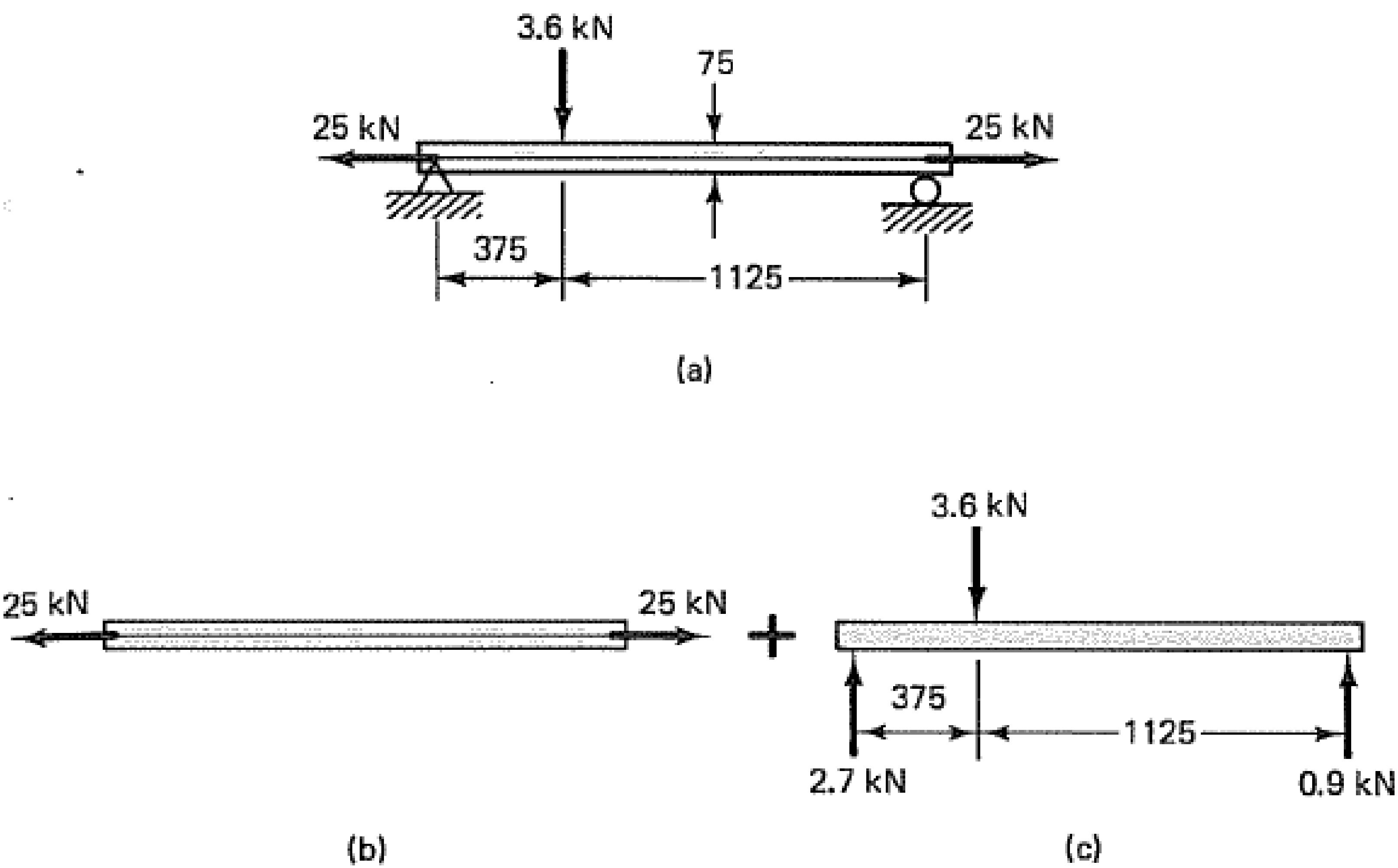


Fig. 6-39

$$\boxed{\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}} \qquad (6\text{-}45)$$

Elastic Bending with Axial Loads – Example Problem

A 50 by 75 mm, 1.5 m long elastic bar of negligible weight is loaded as shown in mm in Fig. 6-40(a). Determine the maximum tensile and compressive stresses acting normal to the section through the beam.



Elastic Bending with Axial Loads – Example Problem

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 75} = 6.67 \text{ MPa} \qquad \text{(tension)}$$

$$\sigma = \frac{Mc}{I} = \frac{6M}{bh^2} = \frac{6 \times 1.013 \times 10^6}{50 \times 75^2} = \pm 21.6 \text{ MPa}$$

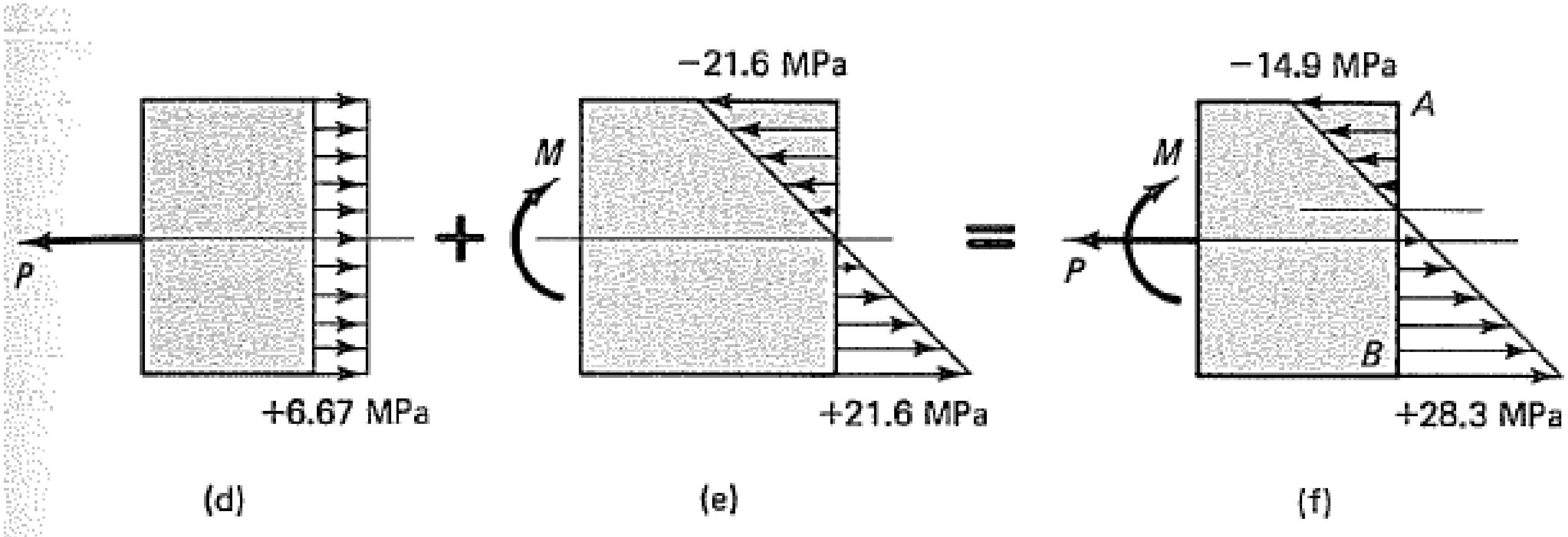


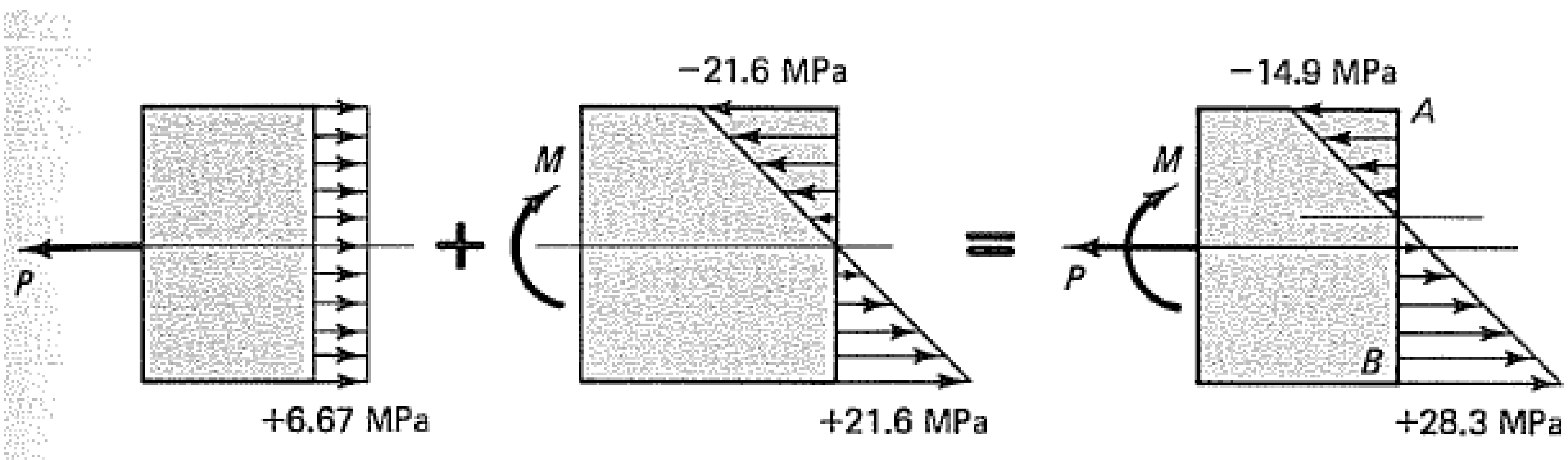
Fig. 6-40

Elastic Bending with Axial Loads – Example Problem

These stresses act normal to the section of the beam and decrease linearly toward the neutral axis as in Fig. 6-40(e). Then, to obtain the compound stress for any particular element, bending stresses must be added algebraically to the direct tensile stress. Thus, as may be seen from Fig. 6-40(f), at point A, the resultant normal stress is 14.9 MPa compression, and at B, it is 28.3 MPa tension. Side views of the stress vectors as commonly drawn are shown in the figure.

Although in this problem, the given axial force is larger than the transverse force, bending causes higher stresses. However, the reader is cautioned not to regard slender compression members in the same light.

Note that in the final result, the line of zero stress, which is located at the centroid of the section for flexure, moves upward. Also note that the local stresses, caused by the concentrated force, which act normal to the top surface of the beam, were not considered. Generally, these stresses are treated independently as local bearing stresses.



(d) (e) (f) Fig. 6-40

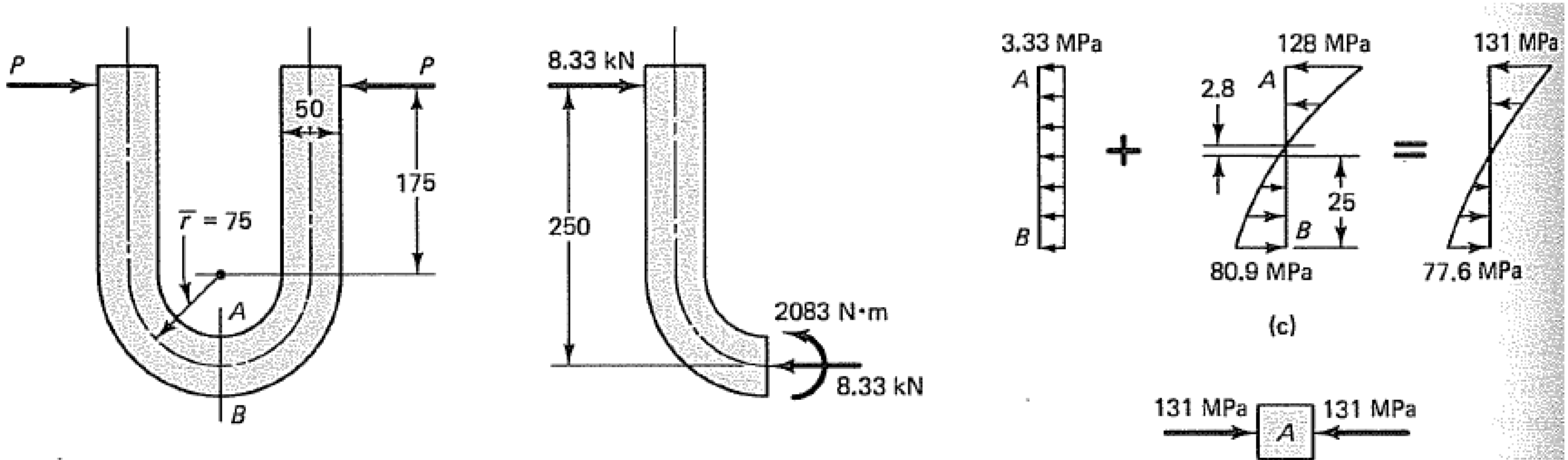
Elastic Bending with Axial Loads – Example Problem

A 50 by 50 mm elastic bar bent into a U shape, as in Fig. 6-41(a), is acted upon by two opposing forces P of 8.33 kN each. Determine the maximum normal stress occurring at section A–B.

Solution

The section to be investigated is in the curved region of the bar, but this makes no essential difference in the procedure. First, a segment of the bar is taken as a free-body, as shown in Fig. 6-41(b). At section A–B, the axial force, applied at the centroid of the section, and the bending moment necessary to maintain equilibrium are determined. Then, each element of the force system is considered separately. The stress caused by the axial forces is

$$\sigma = \frac{P}{A} = \frac{8.33 \times 10^3}{50 \times 50} = 3.33 \text{ MPa} \qquad \text{(compression)}$$



Elastic Bending with Axial Loads – Example Problem

EXAMPLE 6-19

Consider a tapered block having a rectangular cross section at the base, as shown in Figs. 6-42(a) and (b). Determine the maximum eccentricity e such that the stress at B caused by the applied force P is zero.

Solution

In order to maintain applied force P in equilibrium, there must be an axial compressive force P and a moment Pe at the base having the senses shown. The stress caused by the axial force is $\sigma = -P/A = -P/bh$, whereas the largest tensile stress caused by bending is $\sigma_{max} = Mc/I = M/S = 6Pe/bh^2$, where $bh^2/6$ is the elastic section modulus of the rectangular cross section. To satisfy the condition for having stress at B equal to zero, it follows that

$$\sigma_B = -\frac{P}{bh} + \frac{6Pe}{bh^2} = 0 \qquad \text{or} \qquad e = \frac{h}{6}$$

which means that if force P is applied at a distance of $h/6$ from the centroidal axis of the cross section, the stress at B is just zero. Stress distributions across the base corresponding, respectively, to the axial force and bending moment are shown in Figs. 6-42(c) and (d), and their algebraic sum in Fig. 6-42(e).

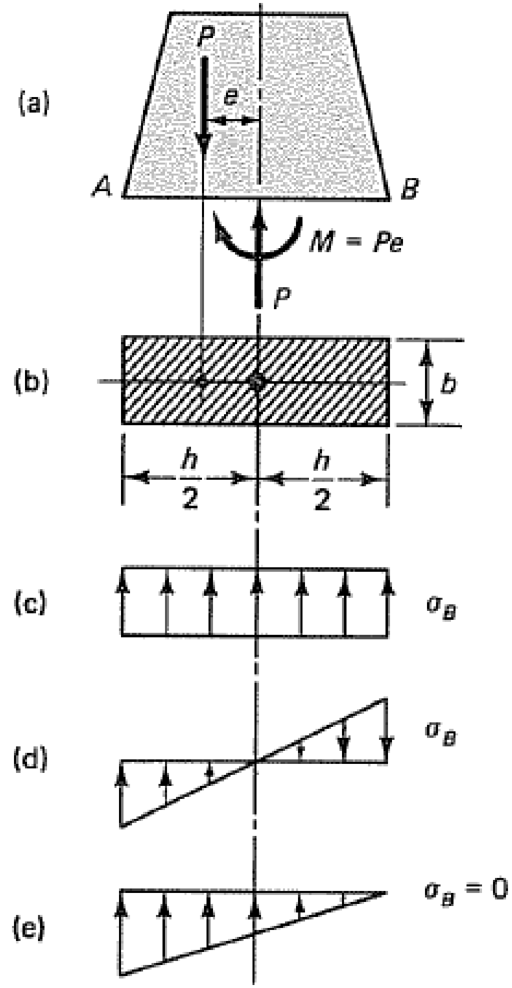


Fig. 6-42 Location of force P causing zero stress at B.

Elastic Bending with Axial Loads – Example Problem

In the above problem, if force P were applied closer to the centroid of the section, a smaller bending moment would be developed at section A–B, and there would be some compression stress at B. The same argument may be repeated for the force acting to the right of the centroidal axis. Hence, a practical rule, much used by the early designers of masonry structures, may be formulated thus: *if the resultant of all vertical forces acts within the middle third of the rectangular cross section, there is no tension in the material at that section*. It is understood that the resultant acts in a vertical plane containing one of the axes of symmetry of the rectangular cross-sectional area.

The foregoing discussion may be generalized in order to apply to any planar system of forces acting on a member. The resultant of these forces may be made to intersect the plane of the cross section, as is shown in Fig. 6-43. At the point of intersection of this resultant with the section, it may be resolved into horizontal and vertical components. If the vertical component of the resultant fulfills the conditions of the former problem, no tension will be developed at point B, as the horizontal component causes only shear stresses. Hence, a more general "middle-third" rule may be stated thus: there will be no tension at a section of a member of a *rectangular* cross section if the resultant of the forces above this section *intersects* one of the axes of symmetry of the section within the middle third.

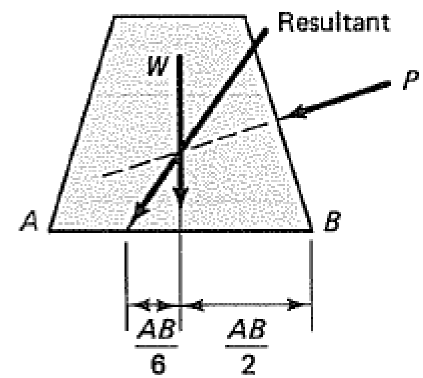


Fig. 6-43 Resultant causing no tension at B.

Kern of Section – Example Problem

Find the zone over which the vertical downward force P_o may be applied to the rectangular weightless block shown in Fig. 6-45(a) without causing any tensile stresses at the section A–B.

Solution

The force $P = -P_o$ is placed at an arbitrary point in the first quadrant of the yz coordinate system shown. Then the same reasoning used in the preceding example shows that with this position of the force, the greatest tendency for a tensile stress exists at A. With $P = -P_o$, $M_z = +P_o y$, and $M_y = -P_o z$, setting the stress at A equal to zero fulfills the limiting condition of the problem. Using Eq. 6-45 allows the stress at A to be expressed as

$$\sigma_A = 0 = \frac{-P_o}{A} - \frac{(P_o y)(-b/2)}{I_{zz}} + \frac{(-P_o z)(-h/2)}{I_{yy}}$$

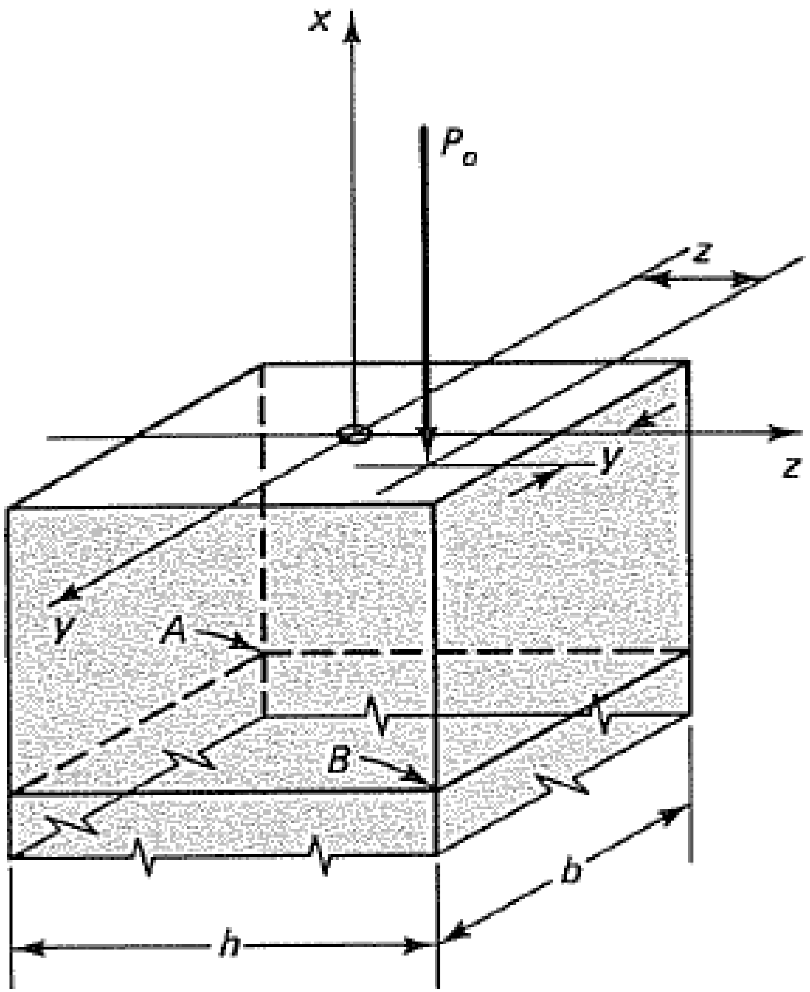
or

$$-\frac{P_o}{A} + \frac{P_o y}{b^2 h/6} + \frac{P_o z}{b h^2/6} = 0$$

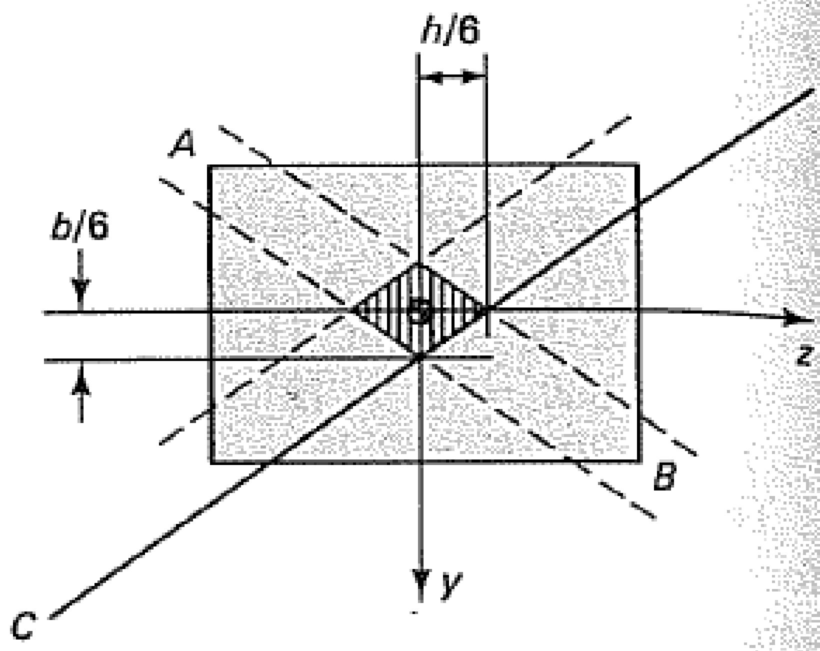
Simplifying,

$$\frac{z}{h/6} + \frac{y}{b/6} = 1$$

Kern of Section – Example Problem



(a)



(b)

Kern of Section – Example Problem

Simplifying, $$\frac{z}{h/6} + \frac{y}{b/6} = 1$$

which is an equation of a straight line. It shows that when $z = 0$, $y = b/6$; and when $y = 0$, $z = h/6$. Hence, this line may be represented by line CD in Fig. 6-45(b). A vertical force may be applied to the block anywhere on this line and the stress at A will be zero. Similar lines may be established for the other three corners of the section; these are shown in Fig. 6-45(b). If force P is applied on any one of these lines or on any line parallel to such a line toward the centroid of the section, there will be no tensile stress at the corresponding corner. Hence, force P may be applied anywhere within the ruled area in Fig. 6-45(b) without causing tensile stress at any of the four corners or anywhere else. This zone of the cross-sectional area is called the *kern* of a section. By limiting the possible location of the force to the lines of symmetry of the rectangular cross section, the results found in this example verify the "middle-third" rule discussed in Example 6-19.

KERN The **kern** of a **section** is the region in which compressive point load may be applied without producing any tensile stress on the cross **section**.

Success Principles

- **Most people fail in life because they MAJOR in MINOR things.**
- **People who succeed at the highest level are not LUCKY. They are doing something differently than everyone else.**
- **Your income right now is a result of your standards. It is not the industry, it is not the economy.**
- **Whatever you hold in your mind on a consistent basis is exactly what you will experience in your life.**
- **Stop being afraid of what could go wrong and start getting EXCITED about what could go RIGHT.**
- **Why live an ordinary life, when you can live an extraordinary one.**
- **Live life fully while you are here. Experience everything. Take care of yourself and your friends. Have fun, be crazy, be weird. Go out and screw up! You are going to anyway, so you must as well enjoy the process. Take the opportunity to learn from your mistakes. Don't try to be perfect; just be an excellent example of being human.**

Courtesy: Anthony Robbins