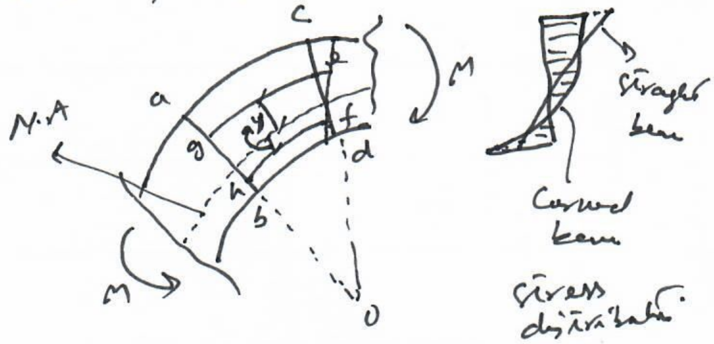
Module # 9
Curved beams

Members subjected to bending are not always straight as hooks of cranes. If members are sharply curved then stress distribution will be different as calculated by formula $\sigma = \frac{My}{I}$

A sharply curved beam subjected to bending is shown in Fig.



Let us assume two fibers ge and hf are equidistance from N.A. So change in length will be same in both the fibers.

$\delta_e = \delta_f$

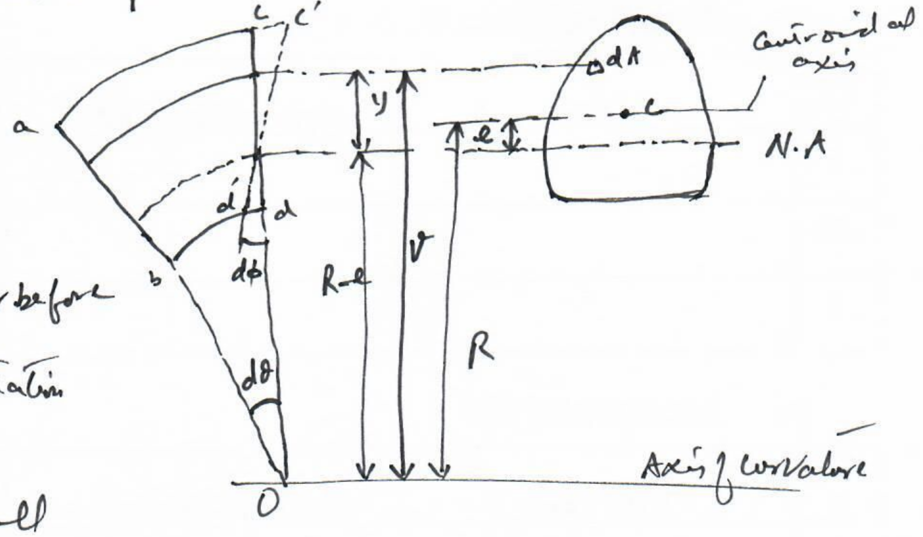
We know that $\delta = \frac{PL}{AE} = \frac{\sigma L}{E}$

$\frac{\sigma_e L_e}{E} = \frac{\sigma_f L_f}{E} \Rightarrow \sigma_e L_e = \sigma_f L_f$

Since L_e is greater than L_f so σ_e will be less than σ_f

So stress distribution is non-linear as shown in the Fig above. As stress distribution is non-linear so there can not be a balance between tensile & compressive forces if N.A passes through the centroid of the section. Hence Neutral axis will shift towards the axis of curvature O.

To derive the stress expression consider the following Fig.



ab and cd represent two adjacent sections of a curved beam. Let these two sections subtends an angle $d\theta$ at the axis of curvature before bending.

$d\phi$ is the angle of rotation of cd relative to ab.

Let y is distance of a small element dA from N.A. Let e is the shift of N.A from centroidal axis. R represent the radius of curvature of the beam.

Total elongation of the fiber at distance y from N.A is $y\,d\phi$. The original length of this fiber is $(R-e+y)\,d\theta$. So strain is

$\epsilon = \frac{\delta}{L} = y\,d\phi / (R-e+y)\,d\theta$

Since $\sigma = E\epsilon$ So $\sigma = \frac{E\,d\phi}{d\theta} \cdot \frac{y}{R-e+y}$ — A

Force in the small element $dA = \sigma\, dA$

So $\sigma\, dA = \frac{E\,d\phi}{d\theta} \cdot \frac{y\,dA}{R-e+y}$

For equilibrium purposes sum of tensile and compressive forces must be equal to zero. So if we integrate dA to entire area then sum force summation must be equal to zero.

$$\int \sigma\, dA = \frac{E\,d\phi}{d\theta} \int \frac{y \cdot dA}{R-e+y} = 0 \Rightarrow \int \frac{y\,dA}{R-e+y} = 0 \quad \text{— (1)}$$

equating the applied bending moment to the resisting moment gives

$$M = \int \sigma\, dA \cdot y = \frac{E\,d\phi}{d\theta} \int \frac{y^2\,dA}{R-e+y} \quad \text{— (2)}$$

Let me ~~evaluate~~ evaluate integral $\int \frac{y^2\,dA}{R-e+y}$

$$\int \frac{y^2\,dA}{R-e+y} = \int \frac{y \cdot y\,dA}{R-e+y} \qquad \therefore \boxed{(R-e+y)-(R-e) = y \text{ — (3)}}$$

Put value of one y from (3) in above

$$= \int \frac{[(R-e+y)-(R-e)] \cdot y\,dA}{R-e+y} = \int \cancel{\left[\frac{1-(R-e)}{R-e+y}\right]}$$

$$= \int \left[1 - \frac{(R-e)}{(R-e+y)}\right] y\,dA$$

$$= \int y\,dA - \int \frac{(R-e)\,y\,dA}{R-e+y} = \int y\,dA - (R-e)\int \frac{y\,dA}{R-e+y}$$

From Ist Integral $\int y\,dA$ Ist moment of area about N.A.

So $\int y\,dA = Ae$

From equation (1) 2nd Integral is equal to zero

So $\int \frac{y^2\,dA}{R-e+y} = Ae$ Put this value in (2)

$$M = \frac{E\,d\phi}{d\theta} \cdot Ae \Rightarrow \frac{E\,d\phi}{d\theta} = \frac{M}{Ae}$$

Substitute this value of $E\frac{d\phi}{d\theta}$ in (A) we get

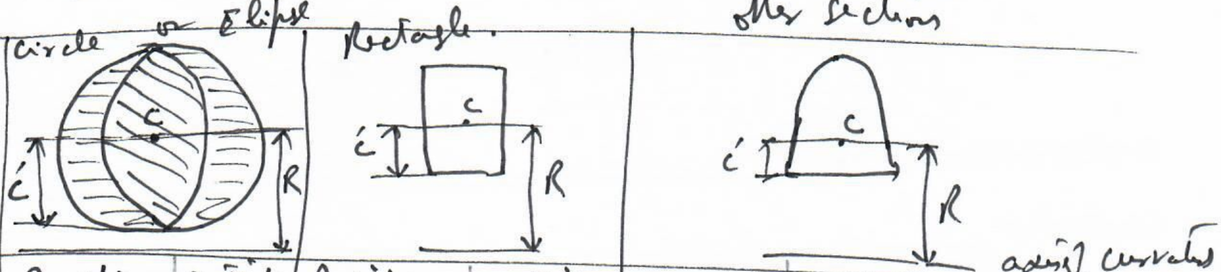
$$\boxed{\sigma = \frac{M}{Ae} \cdot \frac{y}{(R-e+y)}}$$

since $R-e+y$ is distance of exploratory element or fiber from axis of curvature If we call it $V = R-e+y$

So $\boxed{\sigma = \frac{M}{Ae} \times \frac{y}{V}}$

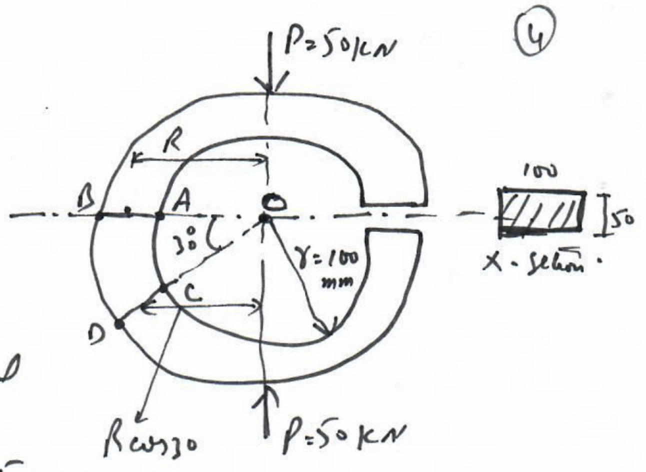
Now instead of using this relation $\sigma = \frac{M}{Ae}\frac{y}{v}$ two researchers B.J. Wilson & J.F. Quereau realize that calculating value of e is difficult hence use of this equation (which can be used to calculate stresses in curved beams) is difficult.

So they compute the stresses in curved beams using the equation and then they compared the results with the flexural formula $\sigma = \frac{Mc}{I}$ ∴ (c is the max. value of y hence gives stresses in extreme fibers).
The amount of stresses more than the straight beam is compensated by using a multiplying factor K.
So according to them stresses in curved beams can be computed by $\boxed{\sigma = K\frac{Mc}{I}}$ where K factor can be noted from the table below.
This table gives values against R/c values where R is radius of curvature and c is extreme fiber distance from the N.A.
They also noted that if R/c is greater than 20 then K is almost unity.
Correction factor K for curved beams.



R/c'	Circle or Ellipse Inside	Circle or Ellipse Outside	Rectangle Inside	Rectangle Outside	Other sections Inside	Other sections Outside
1.2	3.41	0.54	2.89	0.57		
1.4	2.4	0.6	2.13	0.67		
1.6	1.96	0.65	1.79	0.67		
1.8	1.75	0.68	1.63	0.7		
2.0	1.62	0.71	1.52	0.73	1.63	0.73
3.0	1.33	0.79	1.3	0.81	1.36	0.81
4.0	1.23	0.84	1.2	0.85	1.25	0.86
6.0	1.14	0.89	1.12	0.90	1.16	0.9
8.0	1.10	0.91	1.09	0.92	1.12	0.93
10.0	1.08	0.93	1.07	0.94	1.10	0.94
20.0	1.03	0.97	1.04	0.96	1.05	0.95

<u>Num</u> 1. The circular link shown in Fig has a rectangular section 100×50 mm. Compute the stresses at A & B and at C & D. By both formula.



<u>Sol</u>: The radius of curvature R for centroidal axis is 100 + 50 = 150 mm.
C is distance from A to centroidal axis = 50

So $\frac{R}{C} = \frac{150}{50} = 3$

K from Table for ~~...~~ rectangular section is $K_i = 1.3$, $K_o = 0.81$

So $\sigma_A = \frac{K_i MC}{I} = \frac{1.3 \times 50 \times 1000 \times 150 \times 50}{50 \times 100^3/12} =$

$$\boxed{\begin{aligned} M &= 150 \times 1000 \times (100+50) \\ &= 7.5 \times 10^6 \text{ N·mm} \end{aligned}}$$

$\sigma_A = -117$ MPa (Comp)

$\sigma_B = \frac{K_o MC}{I} = \frac{0.81 \times 7.5 \times 10^6 \times 50}{50 \times 100^3/12} = +72.9$ MPa (Ten).

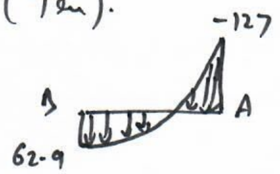
Since load P also Produce a direct compressive stress at A-B

So $\sigma_a = -\frac{P}{A} = \frac{50 \times 1000}{100 \times 50} = -10$ MPa (comp).

Net stress at ~~...~~ A : $\sigma_A = -117 - 10 = -127$ MPa (Comp)

" " " B = $\sigma_B = +72.9 - 10 = 62.9$ MPa (Ten).

So stress Profile at A-B —



<u>For stresses ~~in~~ at C & D</u>

Prependicular distance of mid of C-D from line of P is $R\cos30$

So Moment $M = P \times R\cos30 = 50 \times 1000 \times 150 \times \cos30 = 6.5 \times 10^6$ N·mm

$\sigma_C = \frac{K_i MC}{I} = \frac{1.3 \times 6.5 \times 10^6 \times \cancel{150}\ 50}{50 \times 100^3/12} = -101.4$ MPa (Comp)

$\sigma_D = \frac{K_o MC}{I} = \frac{0.81 \times 6.5 \times 10^6 \times 50}{50 \times 100^3/12} = +63.18$ MPa (Ten)

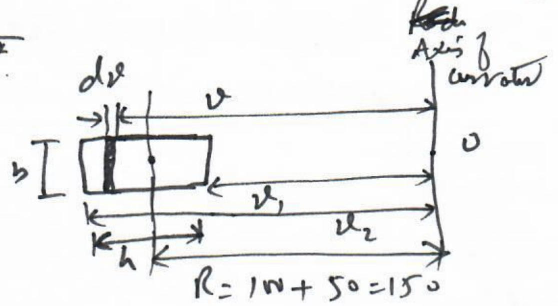
Now direct compressive strength : $-\frac{P}{A}\cos30 = \frac{50 \times 1000}{50 \times 100} \times \cos30 = -\cancel{0.866}$ 8.66 MPa

So Net $\sigma_C = -101.4\ \cancel{4} - 8.66 = -110$ MPa (Comp)

" $\sigma_D = 63.18 - 8.66 = 54.52$ MPa (Ten)

Now using actual formula. $\sigma = \frac{My}{Ae\,v}$

$e = R - \frac{A}{\int \frac{dA}{v}}$



To prove eq^n From (1)

$\int \frac{y\,dA}{R-e+y} = 0$ $\therefore R - e + y = v$
$\quad y = v - (R-e)$

$\int \left(\frac{v - (R-e)}{v} \right) dA = 0 \Rightarrow \int dA - (R-e) \int \frac{dA}{v} = 0$

$A - (R-e)\int \frac{dA}{v} = 0 \quad + R - e = \frac{A}{\int \frac{dA}{v}}$

$e = R - \frac{A}{\int \frac{dA}{v}}$

$e = R - \left(\frac{bh}{\int_{v_1}^{v_2} \frac{b\,dv}{v}} \right) = R - \left(\frac{h}{\log_e \frac{v_2}{v_1}} \right)$

$e = 150 - \left(\frac{100}{\log_e \frac{200}{100}} \right) = 150 - \frac{100}{0.6931} = 5.7$ mm [Note e is shift of N.A]

$\sigma = \frac{My}{Ae\,v}$

$\sigma_A = \frac{50 \times 1000 \times 150 \times (50 - 5.7)}{50 \times 100 \times 5.7 \times 100} =$

v is distance from axis of curvature to the point where stress is measured.

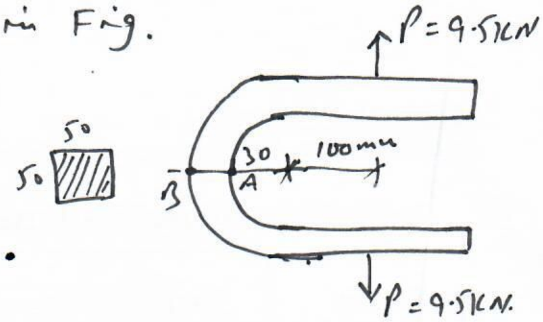
$\sigma_A = -116.58$ (compression)

Net $\sigma_A = -116.58 - 10 = -126.58 \cong -127$ by approximate formula

$\sigma_B = \frac{My}{Ae\,v} = \frac{50 \times 1000 \times 150 \times (50 + 5.7)}{50 \times 100 \times 5.7 \times 200} = 73.29$ Ten

Net $\sigma_B = 73.29 - 10 = 63.29$ Tension $\cong 62.9$ by approximate formula.

Num. A Frame loaded as shown in Fig.



Sol. $\frac{R}{c} = \frac{30+25}{25} = 2.2$

K_i & K_o for 2 are 1.52, 0.73

K_i & K_o for 3 are 1.3, 0.81

So for 2.2 $\quad K_i = 1.52 - \frac{1.52-1.3}{5} = 1.476$

$K_o = 0.73 + \frac{0.81-0.73}{5} = 0.746$

$M = P \times R = 9.5 \times 1000 \times (100+30+25) = 1.473 \times 10^6$ N·mm

$\sigma_A = \frac{P}{A} + K_i \frac{My}{I} = \frac{9500}{50 \times 50} + \frac{1.476 \times 1.473 \times 10^6 \times 25}{50^4/12} = 108.2$ MPa (T)

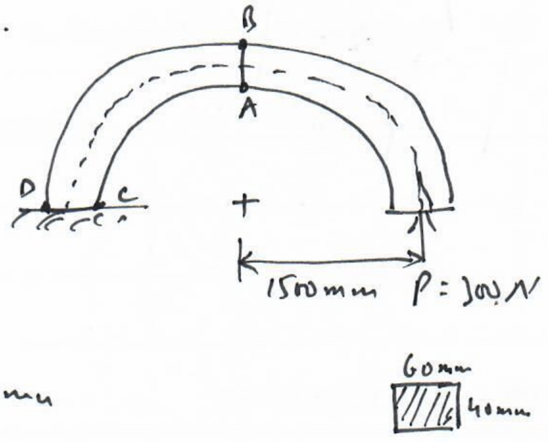
$\sigma_B = \frac{P}{A} - K_o \frac{My}{I} = \frac{9500}{50 \times 50} - \frac{0.746 \times 1.473 \times 10^6 \times 25}{50^4/12} = -48.95$ MPa (C)

Actual stresses $\sigma_A = 106.2$ MPa $\quad$ error about 2%

$\sigma_B = -49.3$ MPa $\quad$ error about 1%

So approximate solution is acceptable.

Num A steel Rod is loaded as shown in Fig. Calculate stresses at Points A, B, C & D.



$\frac{R}{c} = \frac{1500}{30} = 50$

~~Re~~ Approximate formula says that when $\frac{R}{c} > 20$ calculate stress as straight beam & $K = 1$

$M_{AB} = PR = 300 \times 1500 = 450000$ N·mm

$\sigma_A = -\sigma_B = \frac{My}{I} = \frac{450000 \times 30}{40 \times 60^3/12} = 18.75$ MPa

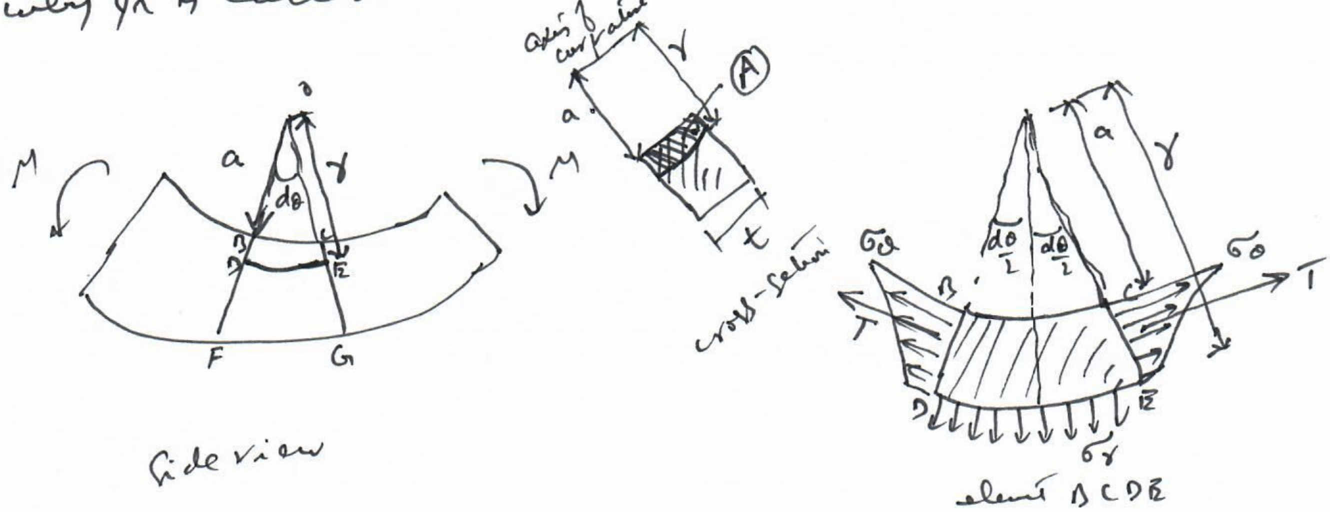
$\sigma_C = -\sigma_D = \frac{300 \times 3000 \times 30}{40 \times 60^3/12} = 37.5$ MPa

Actual values $\sigma_A = 19.32$ (T) & $\sigma_B = -18.79$ (C)

" " $\sigma_C = 38.76$ & $\sigma_D = -37.46$

~~The stress,~~ Radial stresses in curved beams.

The stresses calculated in the previous pages is called circumferential stresses. It may be represented with σ_θ. Now this stress due to equilibrium problem generates another type of stress which acts along the radius of curvature that is why it is called radial stress & may be denoted by σ_r.



Now to establish equilibrium in radial direction.

$$\Sigma F_r = 0$$

$$-\sigma_r \cdot t \cdot r\,d\theta + 2T \sin\frac{d\theta}{2} = 0 \qquad \because \sin\frac{d\theta}{2} = \frac{d\theta}{2}$$

$$-\sigma_r\, t\, r\, d\theta + T \cdot d\theta = 0$$

$$d\theta\,(T - \sigma_r \cdot t\, r) = 0 \quad \Rightarrow \quad \sigma_r = \frac{T}{t\,r} \quad -(1)$$

equation one can be used to find radial stress where T is circumferential force developed in beam, t is thickness of beam at ~~at~~ exploratory section and r is distance from center of curvature to exploratory point.

Since σ_θ ~~is~~ has hyperbolic profile hence calculation of T is not easy.

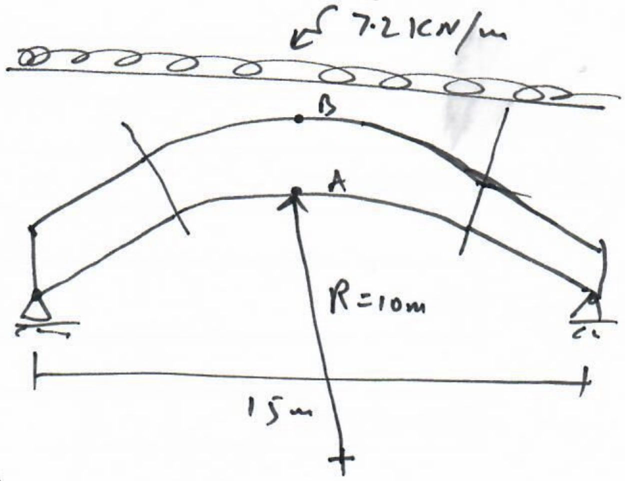
$$T = \int_a^r \sigma_\theta \cdot dA$$

To avoid this integral AITC (American Institute of Timber Construction) has proposed a simple formula for radial stress which is valid for rectangular cross section only:

$$\sigma_r = \frac{3M}{2R_m\,bd}$$

$\because$ M is Moment
R_m is mean radius of curvature
b & d are width & depth of rectangular beam

Num: A Timber beam shown in Fig is used in a roof system. The beam has a simple span of 15m and middle half of beam has curvature of 10m.



Determine the max. circumferential and radial stresses in the beam

Sol:

$b = 130\text{mm}, \quad d = 800\text{mm}$

$$I = \frac{130 \times 800^3}{12} = 55.47 \times 10^8 \text{mm}^4$$

$$R_m = R + \frac{d}{2} = 10 + \frac{0.8}{2} = 10.4\text{m}$$

Radial stress

$$\sigma_r = \frac{3M}{2R_m bd} \qquad \therefore M_{max} = \frac{wl^2}{8} = \frac{7.2 \times 15^2}{8} = 202.5 \text{ KN.m}$$

$$\sigma_r = \frac{3 \times 202.5 \times 10^6}{2 \times 10.4 \times 1000 \times 130 \times 800} = 0.281 \text{ MPa}$$

(Actual formula would have given 0.292 MPa less than 3% error).

Circumferential stress

$$\sigma_\theta = \frac{My}{AeV} \qquad \therefore e = R - \frac{h}{\log_e \frac{r_2}{r_1}} = 10.4 - \frac{0.8}{\log_e \frac{10.8}{10}} = 5.13\text{mm}$$

$$\sigma_{\theta_A} = \frac{202.5 \times 10^6 \times (400 + 5.13)}{130 \times 800 \times 5.13 \times 10 \times 1000} = 15.37 \text{ MPa.} \quad [\text{Tension at inner face}]$$

$$\sigma_{\theta_B} = \frac{202.5 \times 10^6 \times (400 - 5.13)}{130 \times 800 \times 5.13 \times 10.8 \times 1000} = 13.87 \text{ MPa (Compression)}$$

Approximate solution for σ_θ $\quad \frac{R}{C} = \frac{10.4 \times 1000}{400} = 26$

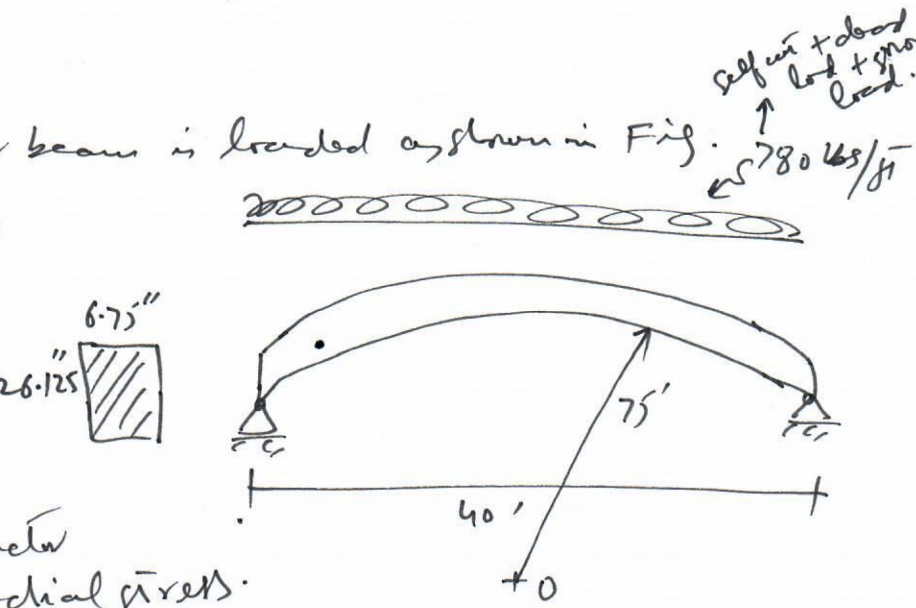
Since $\frac{R}{C}$ is greater than 20 so straight beam formula will be used

$$\sigma_\theta = \frac{My}{I} = \frac{202.5 \times 10^6 \times 400}{55.47 \times 10^8} = \pm 14.6 \text{ MPa}$$ $\boxed{\text{error is about 5\%}}$

Num: A Timber beam is loaded as shown in Fig.

Radius of curvature of Inner face is 75'.

If the beam can bear 82.8 psi Tension in radial direction. Calculate factor of safety against radial stress.



$$M = \frac{wl^2}{8} = \frac{780 \times 40^2}{8} = 156000 \text{ lb.ft}$$

Sol: $b = 6.75''$, $d = 26.125''$, $I = \frac{6.75 \times 26.125^3}{12} \cancel{\times 12^4} = 208 \times 10^6 \text{ in}^4$

$$R_m = R + \frac{d}{2} = 75 \times 12 + \frac{26.125}{2} = 913.063''$$

$$\sigma_r = \frac{3M}{2 R_m b d} = \frac{3 \times 156000 \times 12}{2 \times 913.063 \times 6.75 \times 26.125} = \boxed{17.44 \text{ psi}}$$

$$F.O.S = \frac{\text{Allowable stress}}{\text{Actual stress}}$$

$$= \frac{82.8}{17.44} = \boxed{4.75}$$