Mechanics of Solids-2

Thick Walled Cylinders

Lecture # 2

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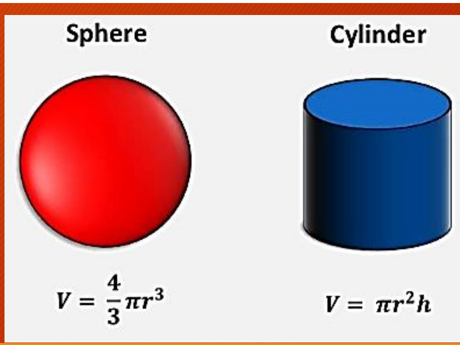
Cylindrical and Spherical Pressure Vessels

Cylindrical Pressure Vessel

Spherical Pressure Vessel



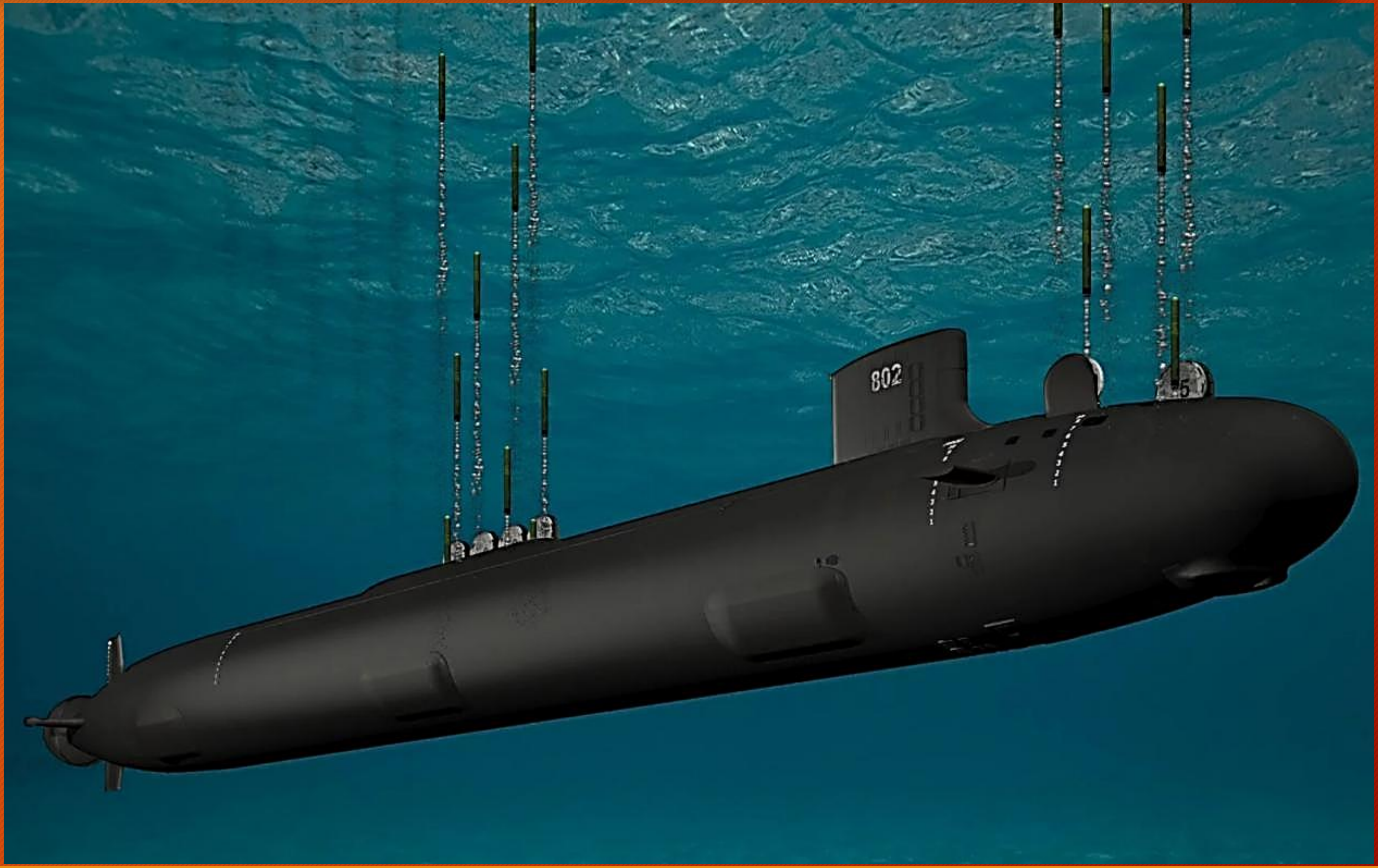
Thick Walled Pressure Vessels - Applications

Applications

- $p_i = 0$
  - Submarine
  - Vacuum chamber
  - Shrink fit
  - Buried pipe
- $p_o = 0$
  - Gun barrel
  - Liquid- or gas-carrying pipe
  - Hydraulic cylinder
  - Gas storage tank

A vessel can be a ship, a container for holding liquids, or a tube that transports blood throughout your body.

Thick Walled Pressure Vessels - Applications
Submarines



Thick Walled Pressure Vessels - Applications
Vacuum Chamber

Thick Walled Pressure Vessels - Applications
Shrink Fit

Shrink-fitting is a technique in which an interference fit is achieved by a relative size change after assembly. Wikipedia

Thick Walled Pressure Vessels - Applications
Buried Pipes

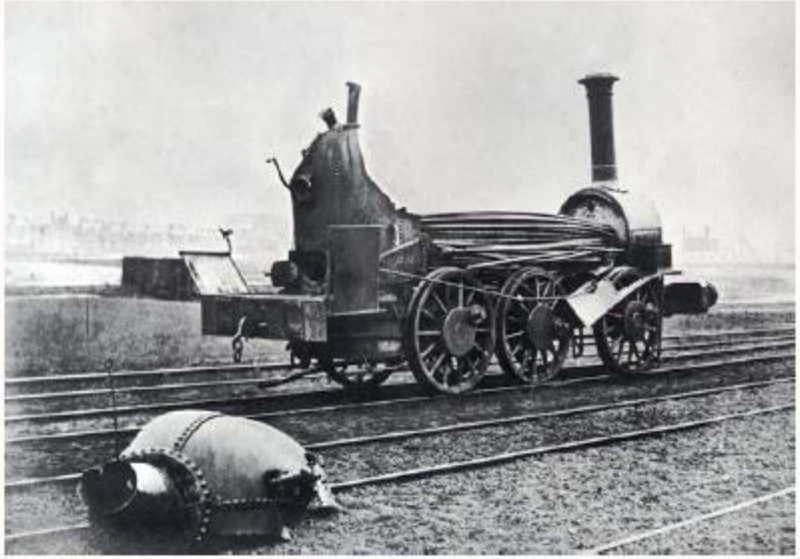
Thick Walled Pressure Vessels - Applications
Gun Barrels

Thick Walled Cylinders - Applications
Cylinders, Gas Pipelines, and Storage Tanks

Thick-walled cylinders see wide use in a number of challenging applications in the engineering, oil and gas, structural, petrochemical, nuclear and pressure vessel industries. The thick walls offer increased resistance to pressure or aggressive media, a key concern for operators within these sectors.

Thick Walled Cylinders - Failures in History

Credits: www.pveng.com



Credits: www.wikipedia.org

Thick Walled Cylinders - Differences

Thin Cylinders	Thick Cylinders
$\frac{D_i}{t} > 20$	$\frac{D_i}{t} < 20$
$\sigma_L, \sigma_r, \sigma_\theta$ all are constant	σ_L is constant σ_r, σ_θ vary with radius
σ_r is negligible	

THIN AND THICK CYLINDERS

INTRODUCTION:

In many engineering applications, cylinders are frequently used for transporting or storing of liquids, gases or fluids.

Eg: Pipes, Boilers, storage tanks etc.

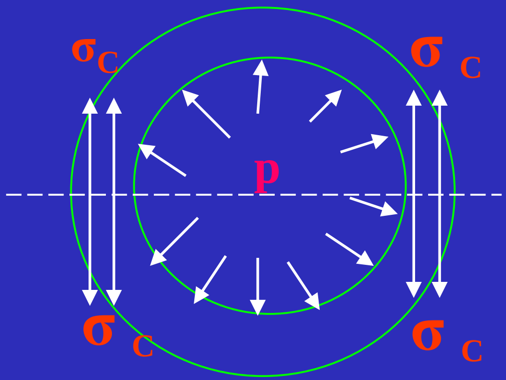
These cylinders are subjected to fluid pressures. When a cylinder is subjected to a internal pressure, at any point on the cylinder wall, three types of stresses are induced on three mutually perpendicular planes.

They are,

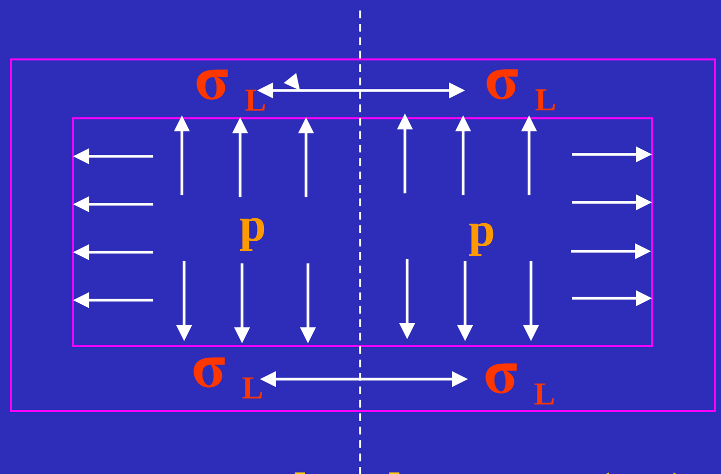
1. Hoop or Circumferential Stress (σ_C) – This is directed along the tangent to the circumference and tensile in nature. Thus, there will be increase in diameter.

2. Longitudinal Stress (σ_L) – This stress is directed along the length of the cylinder. This is also tensile in nature and tends to increase the length.

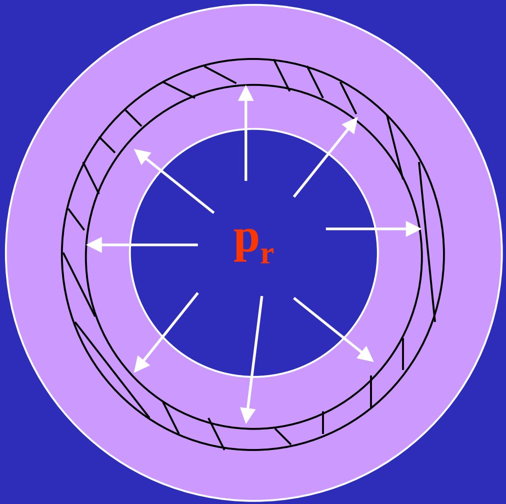
3. Radial pressure (p_r) – It is compressive in nature. Its magnitude is equal to fluid pressure on the inside wall and zero on the outer wall if it is open to atmosphere.



1. Hoop Stress (σ_C)

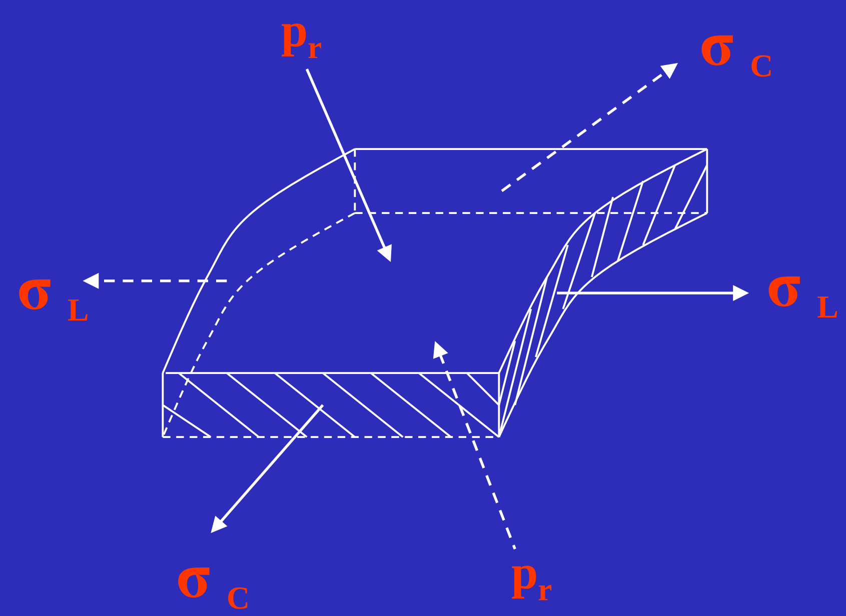


2. Longitudinal Stress (σ_L)



3. Radial Stress (p_r)

Element on the cylinder wall subjected to these three stresses



THICK WALLED CYLINDERS

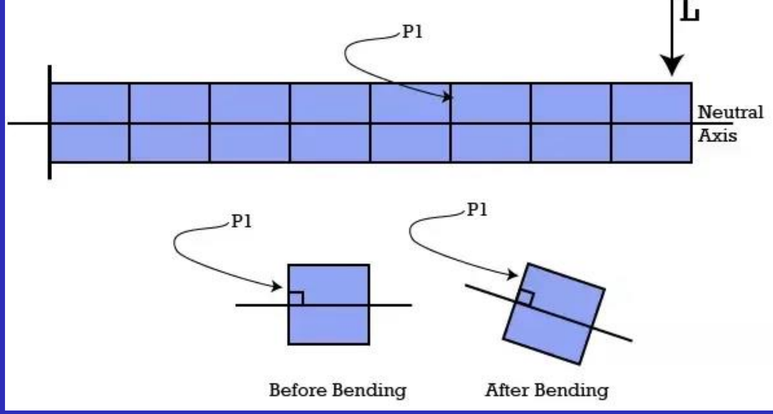
INTRODUCTION:

The thickness of the cylinder is large compared to that of thin cylinder.

i. e., in case of thick cylinders, the metal thickness ‘t’ is more than ‘d/20’, where ‘d’ is the internal diameter of the cylinder.

Magnitude of radial stress (p_r) is large, and hence it cannot be neglected. The circumferential stress is also not uniform across the cylinder wall. The radial stress is compressive in nature, and circumferential and longitudinal stresses are tensile in nature. Radial stress and circumferential stresses are computed by using ‘Lame’s equations’.

<u>LAME'S EQUATIONS</u> (Theory) :



ASSUMPTIONS:

1. Plane sections of the cylinder normal to its axis remain plane and normal even under pressure.

2. Longitudinal stress (σ_L) and longitudinal strain (ε_L) remain constant throughout the thickness of the wall.

3. Since, longitudinal stress (σ_L) and longitudinal strain (ε_L) are constant, it follows that the difference in the magnitude of hoop stress and radial stress (p_r) at any point on the cylinder wall is a constant.

Generalized Hooks Law (Tensile Force Only)

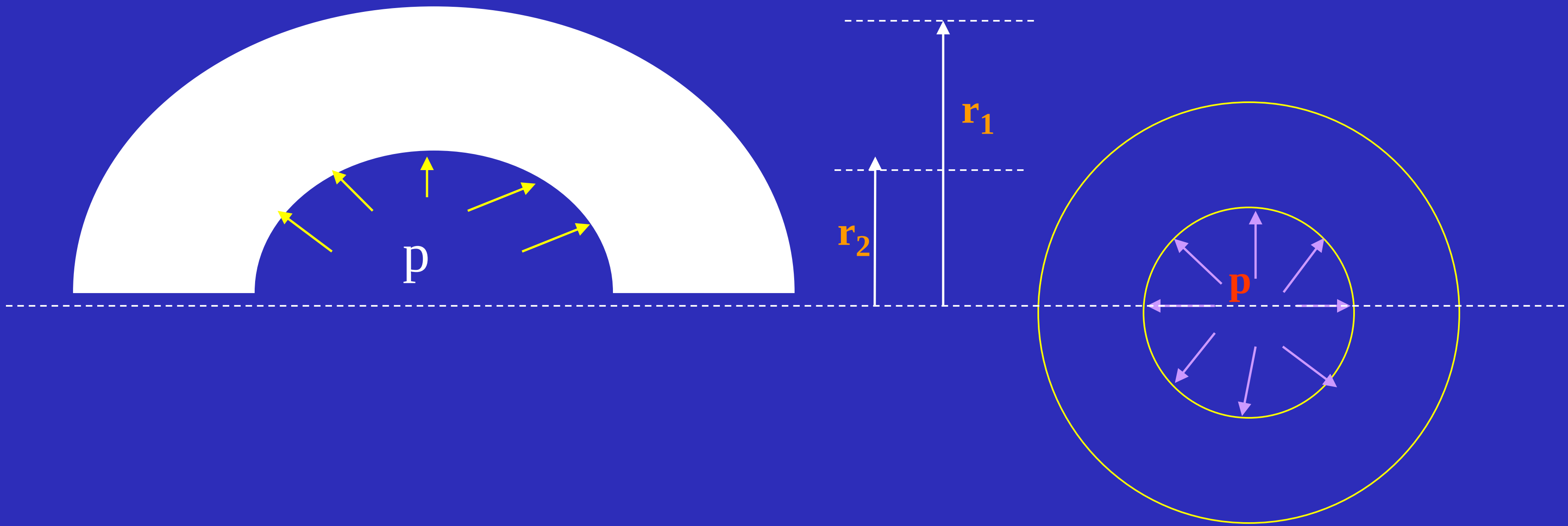
$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

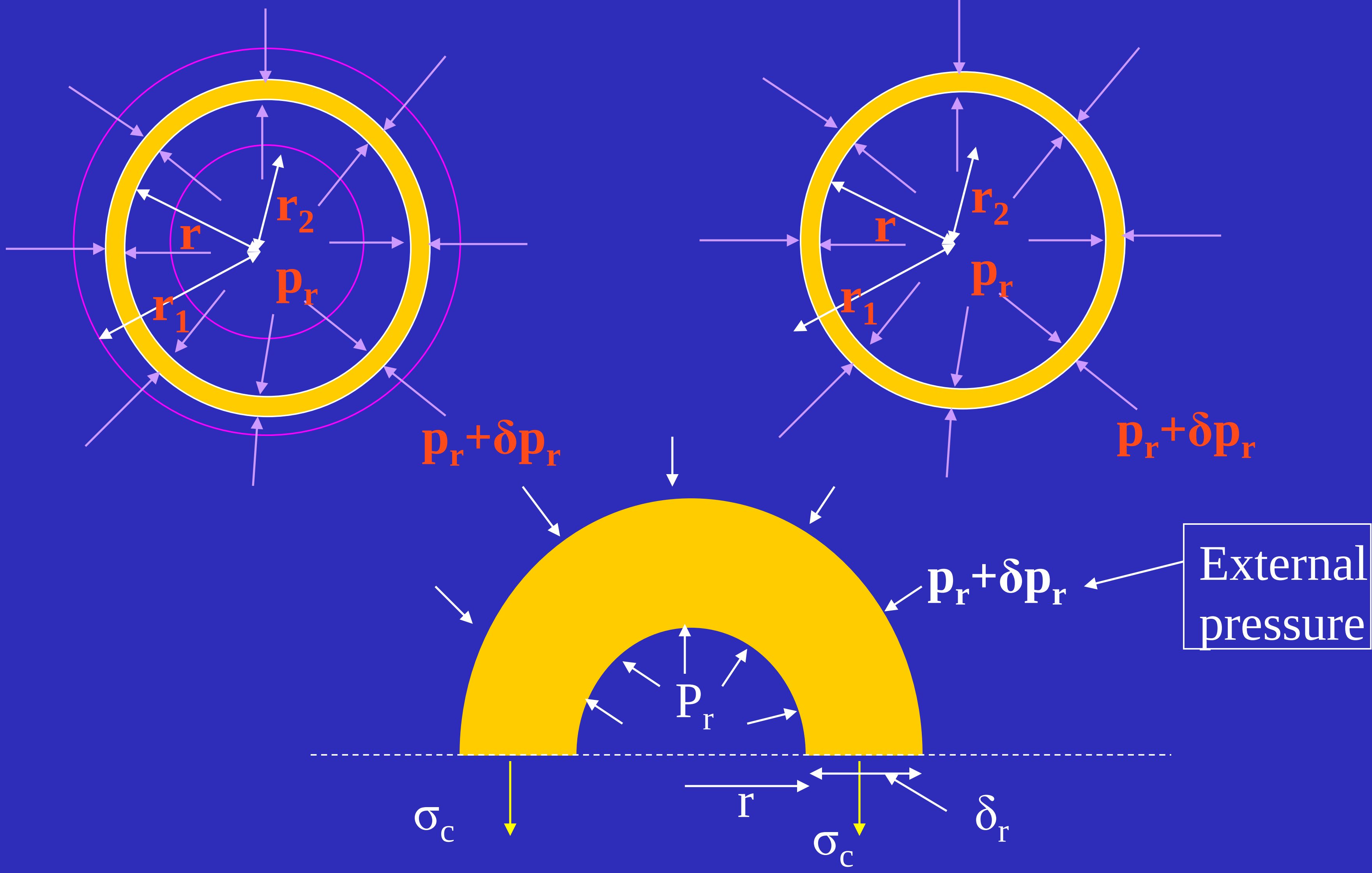
4. The material is homogeneous, isotropic, and elastic i.e. obeys Hooke's law (stresses are within proportionality limit).

LAME'S EQUATIONS FOR RADIAL PRESSURE AND CIRCUMFERENTIAL STRESS



Consider a thick cylinder of external radius r_1 and internal radius r_2, containing a fluid under pressure 'p' as shown in the fig.

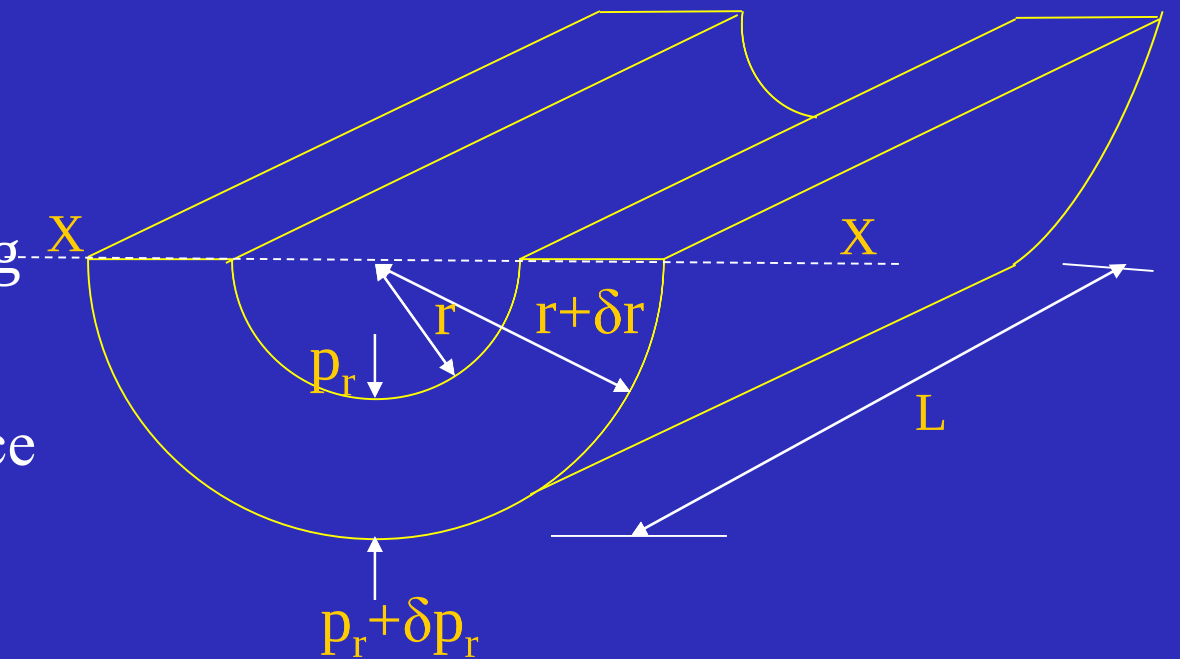
Let 'L' be the length of the cylinder.



Consider an elemental ring of radius ‘r’ and thickness ‘δ_r’ as shown in the above figures. Let p_r and $(p_r + \delta p_r)$ be the intensities of radial pressures at inner and outer faces of the ring.

Consider the longitudinal section XX of the ring as shown in the fig.

The bursting force is evaluated by considering the projected area, '$2\times r\times L$' for the inner face and '$2\times(r+\delta_r)\times L$' for the outer face .



The Net Bursting Force, P $= p_r\times 2\times r\times L - (p_r+\delta p_r)\times 2\times(r+\delta_r)\times L$

$$=(-p_r\times \delta_r - r\times\delta p_r - \delta p_r \times \delta_r)\, 2L$$

Bursting force is resisted by the hoop tensile force developing at the level of the strip i.e.,

$$F_r=\sigma_c\times 2 \times\delta_r\times L$$

Thus, for equilibrium, $P = F_r$

$$(-p_r \times \delta_r - r \times \delta p_r - \delta p_r \times \delta_r)\, 2L = \sigma_c \times 2 \times \delta_r \times L$$

$$-p_r \times \delta r - r \times \delta p_r - \delta p_r \times \delta_r = \sigma_c \times \delta r$$

Neglecting products of small quantities, (i.e., $\delta p_r \times \delta r$)

$$\sigma_c = -p_r - (r \times \delta p_r)/\delta_r \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Since P_r is compressive

Longitudinal strain is constant. Hence we have,

$$\varepsilon_L = \boxed{\frac{\sigma_L}{E} - \mu \times \frac{\sigma_C}{E} + \mu \times \frac{p_r}{E} = \text{constant}}$$

$$\varepsilon_L = \boxed{\frac{\sigma_L}{E} - \frac{\mu}{E}(\sigma_C - p_r) = \text{constant}}$$

Generalized Hooks Law (Tensile Force Only)

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

since σ_L, E and μ are constants, $(\sigma_c - P_r)$ should be constant . Let it be equal to 2a. Thus

$\sigma_c - p_r = 2a$,

i.e., $\sigma_c = p_r + 2a$ ………………..(2)

From (1), $p_r + 2a = -p_r - (r \times \delta p_r) / \delta_r$

$p_r + p_r + 2a = -(r \times \delta p_r) / \delta_r$

i. e.,

$$2(p_r + a) = -r \times \frac{\delta p_r}{\delta_r}$$

$$-2 \times \frac{\delta_r}{r} = \frac{\delta p_r}{(p_r + a)} \text{ ....(3)}$$

Integrating, $(-2 \times \log_e r) + c = \log_e (p_r + a)$

Where c is constant of integration. Let it be taken as $\log_e b$, where 'b' is another constant.

Thus, $\log_e (p_r + a) = -2 \times \log_e r + \log_e b = -\log_e r^2 + \log_e b = \log_e \frac{b}{r^2}$

$$\int 1\,dx = x + c$$

$$\int \frac{1}{x}\,dx = \log|x| + c$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$$

$$\cancel{\log_e}(p_r+a) = \cancel{\log_e}\ \frac{b}{r^2}$$

$$p_r + a = \frac{b}{r^2}$$

Radial Stress, $p_r = \frac{b}{r^2} - a$ ....(4) **(Compressive)**

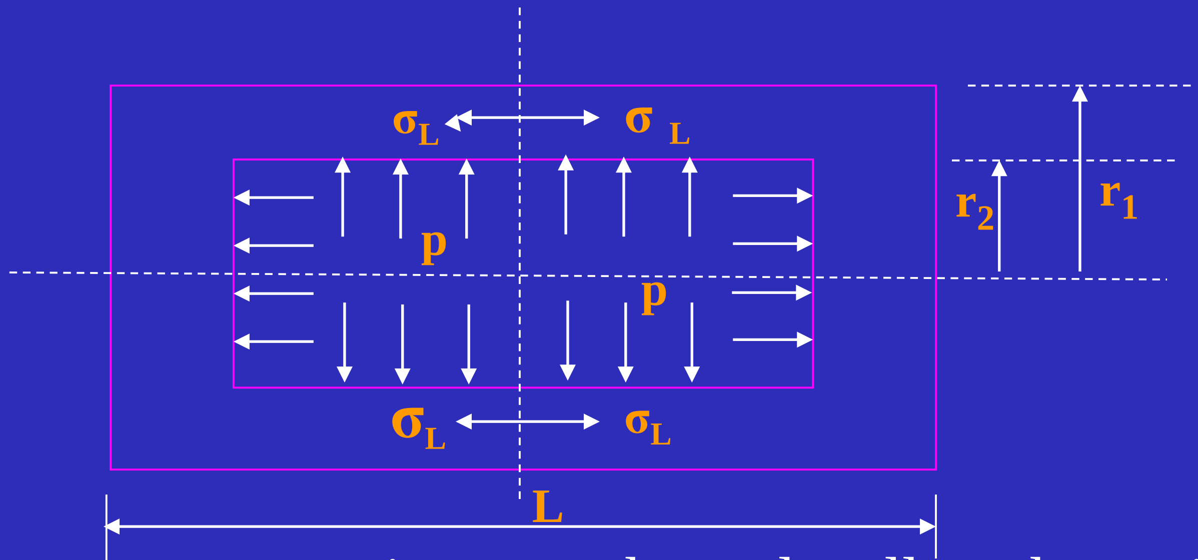
Substituting it in equation 2, we get

Hoop stress, $\sigma_c = p_r + 2a = \frac{b}{r^2} - a + 2a$

Circumferial Stress, $\sigma_c = \frac{b}{r^2} + a$ ....(5) **(Tensile)**

The equations (4) & (5) are known as "Lame's Equations" for radial pressure and hoop stress at any specified point on the cylinder wall. Thus, $r_1 \leq r \leq r_2$.

ANALYSIS FOR LONGITUDINAL STRESS



Consider a transverse section near the end wall as shown in the fig.
Bursting force, $P = \pi \times r_2^2 \times p$

Resisting force is due to longitudinal stress 'σ_L'.

i.e., $F_L = \sigma_L \times \pi \times (r_1^2 - r_2^2)$

For equilibrium, $F_L = P$

$\sigma_L \times \pi \times (r_1^2 - r_2^2) = \pi \times r_2^2 \times p$

Therefore, longitudinal stress,

$$\boxed{\sigma_L = \frac{p \times r_2^2}{(r_1^2 - r_2^2)} \quad \text{(Tensile)}}$$

Important Points:

1. Variations of Hoop stress and Radial stress are parabolic across the cylinder wall.
2. At the inner edge, the stresses are maximum.
3. The value of 'Permissible or Maximum Hoop Stress' is to be considered on the inner edge.
4. The maximum shear stress (σ_{max}) and Hoop, Longitudinal and radial strains (ε_c, ε_L, ε_r) are calculated as in thin cylinder but separately for inner and outer edges.

Analysis Problem

ILLUSTRATIVE PROBLEMS

PROBLEM 1:

A thick cylindrical pipe of external diameter 300mm and internal diameter 200mm is subjected to an internal fluid pressure of 20N/mm^2 and external pressure of 5 N/mm^2. Determine the maximum hoop stress developed and draw the variation of hoop stress and radial stress across the thickness. Show at least four points for each case.

SOLUTION:

External diameter = 300mm. ⇨ External radius, r_1=150mm.

Internal diameter = 200mm. ⇨ Internal radius, r_2=100mm.

Lame's equations:

For Hoop stress, $$\sigma_c = \frac{b}{r^2} + a \quad ....(1)$$

For radial stress, $$p_r = \frac{b}{r^2} - a \quad ....(2)$$

Boundary conditions:

At r =100mm (on the inner face), radial pressure = 20N/mm^2

$$\text{i.e.,} \quad 20 = \frac{b}{100^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly, at r =150mm (on the outer face), radial pressure = 5N/mm^2

$$\text{i.e.,} \quad 5 = \frac{b}{150^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Solving equations (3) & (4), we get a = 7, b = 2,70,000.

Lame's equations are, for Hoop stress, $\sigma_c = \dfrac{2{,}70{,}000}{r^2} + 7 \quad \ldots\ldots\ldots\ldots(5)$

For radial stress, $p_r = \dfrac{2{,}70{,}000}{r^2} - 7 \quad \ldots\ldots\ldots\ldots(6)$

Hoop Stresses will always be tensile.
Radial Stresses will always be compressive.

To draw variations of Hoop stress & Radial stress :

At r =100mm (on the inner face),

Point # 1

$$\text{Hoop stress, } \sigma_c = \frac{2{,}70{,}000}{100^2} + 7 = 34 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2{,}70{,}000}{100^2} - 7 = 20 \text{ MPa (Comp)}$$

At r =120mm,

Point # 2

$$\text{Hoop stress, } \sigma_c = \frac{2{,}70{,}000}{120^2} + 7 = 25.75 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2{,}70{,}000}{120^2} - 7 = 11.75 \text{ MPa (Comp)}$$

At r =135mm,

Point # 3

$$\text{Hoop stress, } \sigma_c = \frac{2{,}70{,}000}{135^2} + 7 = 21.81 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2{,}70{,}000}{135^2} - 7 = 7.81 \text{ MPa (Comp)}$$

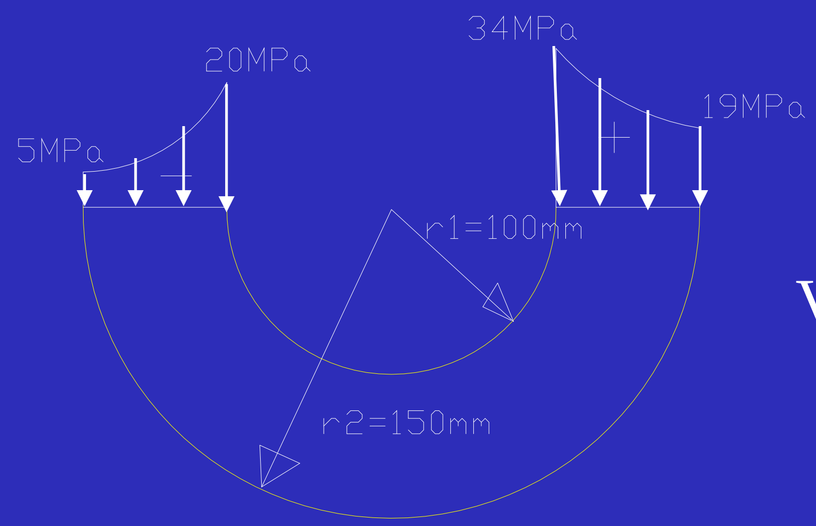
Point # 4

At $r = 150\text{mm}$,

$$\text{Hoop stress, } \sigma_c = \frac{2,70,000}{150^2} + 7 = 19 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2,70,000}{150^2} - 7 = 5 \text{ MPa (Comp)}$$

Variation of Radial Stress –Comp (Parabolic)



Variation of Hoop Stress-Tensile (Parabolic)

Variation of Hoop stress & Radial stress

PROBLEM 2:

Find the thickness of the metal required for a thick cylindrical shell of internal diameter 160mm to withstand an internal pressure of 8 N/mm^2. The maximum hoop stress in the section is not to exceed 35 N/mm^2.

SOLUTION:

Internal radius, r_2=80mm.

Lame's equations are,

$$\text{for Hoop Stress, } \sigma_c = \frac{b}{r^2} + a \text{ ....(1)}$$

$$\text{for Radial stress, } p_r = \frac{b}{r^2} - a \text{ ....(2)}$$

Boundary conditions are,

at $r = 80\text{mm}$, radial stress $p_r = 8\ \text{N/mm}^2$,

and Hoop stress, $\sigma_C = 35\ \text{N/mm}^2$. ($\because$ Hoop stress is max on inner face)

i.e.,
$$8 = \frac{b}{80^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$
$$35 = \frac{b}{80^2} + a \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Solving equations (3) & (4), we get $a = 13.5$, $b = 1{,}37{,}600$.

$\therefore$ Lame's equations are,
$$\sigma_c = \frac{1{,}37{,}600}{r^2} + 13.5 \quad \ldots\ldots\ldots\ldots(5)$$

and
$$p_r = \frac{1{,}37{,}600}{r^2} - 13.5 \quad \ldots\ldots\ldots\ldots(6)$$

On the outer face, pressure = 0.

i.e., $p_r = 0$ at $r = r_1$.

$$\therefore \quad 0 = \frac{1{,}37{,}600}{r_1^{\,2}} - 13.5$$

$$\therefore \quad r_1 = \underline{100.96\text{mm.}}$$

$$\therefore \quad \text{Thickness of the metal} = r_1 - r_2$$

$$= \underline{20.96\text{mm.}}$$

PROBLEM 3:

A thick cylindrical pipe of outside diameter 300mm and internal diameter 200mm is subjected to an internal fluid pressure of 14 N/mm^2. Determine the maximum hoop stress developed in the cross section. What is the percentage error if the maximum hoop stress is calculated by the equations for thin cylinder?

SOLUTION:

Internal radius, r_2=100mm.

External radius, r_1=150mm

.

Lame's equations:

For Hoop stress, $\sigma_c = \frac{b}{r^2} + a$ ....(1)

For radial pressure, $p_r = \frac{b}{r^2} - a$ ....(2)

Boundary conditions:

At x =100mm $P_r = 14 N/mm^2$

$$\text{i.e.,} \quad 14 = \frac{b}{100^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly, at x =150mm $P_r = 0$

$$\text{i.e.,} \quad 0 = \frac{b}{150^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving, equations (1) & (2), we get a =11.2, b = 2,52,000.

$$\therefore \text{Lame's equation for Hoop stress,} \quad \sigma_r = \frac{22{,}500}{r^2} + 11.2 \quad \ldots\ldots\ldots\ldots(3)$$

Max hoop stress on the inner face (where x=100mm):

$$\sigma_{max} = \frac{252000}{100^2} + 11.2 = \underline{36.4\ \mathrm{MPa.}}$$

By thin cylinder formula, $\sigma_{max} = \dfrac{p \times d}{2 \times t}$

where $D = 200\mathrm{mm},\ t = 50\mathrm{mm}$ and $p = 14\mathrm{MPa}.$

$$\therefore \sigma_{max} = \frac{14 \times 200}{2 \times 50} = \underline{28\mathrm{MPa.}}$$

$$\boxed{\text{Percentage error} = \left(\frac{36.4 - 28}{36.4}\right) \times 100 = \underline{23.08\%}.}$$

PROBLEM 4:

The principal stresses at the inner edge of a cylindrical shell are 81.88 MPa (T) and 40MPa (C). The internal diameter of the cylinder is 180mm and the length is 1.5m. The longitudinal stress is 21.93 MPa (T). Find,

(i) Max shear stress at the inner edge.
(ii) Change in internal diameter.
(iii) Change in length.
(iv) Change in volume.

Take E=200 GPa and μ=0.3.

SOLUTION:

i) Max shear stress on the inner face :

$$\tau_{max} = \frac{\sigma_C - p_r}{2} = \frac{81.88 - (-40)}{2}$$

$$= 60.94 \text{ MPa}$$

ii) Change in inner diameter :

$$\frac{\delta d}{d} = \frac{\sigma_C}{E} - \frac{\mu}{E} \times p_r - \frac{\mu}{E} \times \sigma_L$$

$$= \frac{81.88}{200 \times 10^3} - \frac{0.3}{200 \times 10^3} \times (-40) - \frac{0.3}{200 \times 10^3} \times (21.93)$$

$$= 4.365 \times 10^{-4}$$

$$\therefore \quad \delta d = \underline{+0.078\text{mm.}}$$

iii) Change in Length :

$$\frac{\delta l}{L} = \frac{\sigma_L}{E} - \frac{\mu}{E} \times p_r - \frac{\mu}{E} \times \sigma_C$$

$$= \frac{21.93}{200 \times 10^3} - \frac{0.3}{200 \times 10^3} \times (-40) - \frac{0.3}{200 \times 10^3} \times 81.88$$

$$= 46.83 \times 10^{-6}$$

$$\therefore \quad \delta l = \underline{+0.070\text{mm.}}$$

iv) Change in volume :

$$\frac{\delta V}{V} = \frac{\delta l}{L} + 2 \times \frac{\delta d}{D}$$

$$= 9.198 \times 10^{-4}$$

$$\therefore \quad \delta V = 9.198 \times 10^{-4} \times \left(\frac{\pi \times 180^2 \times 1500}{4} \right)$$

$$= \underline{35.11 \times 10^3} \text{ mm}^3.$$

PROBLEM 5:

Find the max internal pressure that can be allowed into a thick pipe of outer diameter of 300mm and inner diameter of 200mm so that tensile stress in the metal does not exceed 16 MPa if, (i) there is no external fluid pressure, (ii) there is a fluid pressure of 4.2 MPa.

SOLUTION:

External radius, r_1=150mm.
Internal radius, r_2=100mm.

Case (i) – When there is no external fluid pressure:

Boundary conditions:
At r=100mm , $\sigma_c = 16N/mm^2$
At r=150mm , $P_r = 0$

$$\text{i.e.,} \qquad 16 = \frac{b}{100^2} + a \; \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$0 = \frac{b}{150^2} - a \; \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving we get, $a = 4.92$ & $b = 110.77 \times 10^3$

$$\text{so that} \qquad \sigma_c = \frac{110.77 \times 10^3}{r^2} + 4.92 \; \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$p_r = \frac{110.77 \times 10^3}{r^2} - 4.92 \; \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Fluid pressure on the inner face where $r = 100\text{mm}$,

$$p_r = \frac{110.77 \times 10^3}{100^2} - 4.92 = \underline{6.16 \text{ MPa.}}$$

Case (ii) – When there is an external fluid pressure of 4.2 MPa:

Boundary conditions:

At r=100mm , σ_c= 16 N/mm^2

At r=150mm , p_r= 4.2 MPa.

i.e.,
$$16 = \frac{b}{100^2} + a \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$4.2 = \frac{b}{150^2} - a \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving we get, a = 2.01 & b=139.85×10^3

so that
$$\sigma_r = \frac{139.85\times 10^3}{r^2} + 2.01 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$p_r = \frac{139.85\times 10^3}{r^2} - 2.01 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Fluid pressure on the inner face where $r = 100\text{mm}$,

$$p_r = \frac{139.85 \times 10^3}{100^2} - 2.01 = \underline{11.975} \text{ MPa.}$$

PROBLEMS FOR PRACTICE

PROBLEM 1:

A pipe of 150mm internal diameter with the metal thickness of 50mm transmits water under a pressure of 6 MPa. Calculate the maximum and minimum intensities of circumferential stresses induced.

(Ans: 12.75 MPa, 6.75 MPa)

PROBLEM 2:

Determine maximum and minimum hoop stresses across the section of a pipe of 400mm internal diameter and 100mm thick when a fluid under a pressure of 8N/mm^2 is admitted. Sketch also the radial pressure and hoop stress distributions across the thickness.

(Ans: σ_{max}=20.8 N/mm^2, σ_{min}=12.8 N/mm^2)

PROBLEM 3:

A thick cylinder with external diameter 240mm and internal diameter ‘D’ is subjected to an external pressure of 50 MPa. Determine the diameter ‘D’ if the maximum hoop stress in the cylinder is not to exceed 200 MPa.

(Ans: 169.7 mm)

PROBLEM 4:

A thick cylinder of 1m inside diameter and 7m long is subjected to an internal fluid pressure of 40 MPa. Determine the thickness of the cylinder if the maximum shear stress in the cylinder is not to exceed 65 MPa. What will be the increase in the volume of the cylinder? E=200 GPa, μ=0.3. (Ans: t=306.2mm, $\delta v = 5.47\times10^{-3} m^3$)

PROBLEM 5:

A thick cylinder is subjected to both internal and external pressure. The internal diameter of the cylinder is 150mm and the external diameter is 200mm. If the maximum permissible stress in the cylinder is 20 N/mm^2 and external radial pressure is 4 N/mm^2, determine the intensity of internal radial pressure. (Ans: 10.72 N/mm^2)

Assignment # 2: Thick Walled Cylinders Problems

Solve all the problems given in "Problem for Practice".

Due Date: 20th February, 2020.

Quiz # 1: Thin Walled Cylinders - Problems Only

Content Included:

Lecture # 1 – ‘Example Problems’ + Assignment # 1 “Problem for Practice”.

Date: 20th February, 2020.

Quiz # 2: Thick Walled Cylinders - Problems Only

Content Included:

Lecture # 2 – 'Example Problems' + Assignment # 2 'Problem for Practice'

Date: 20th February, 2020.

Thank-you for Listening!

To get wisdom, 'listen the unheard'. (Shad)